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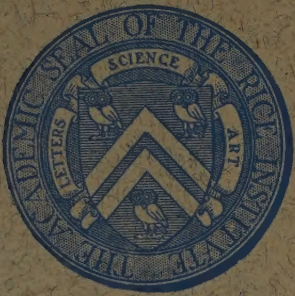

MONOGRAPH IN MATHEMATICS

RIEMANN SURFACES AND ASYMPTOTIC VALUES ASSOCIATED WITH REAL ENTIRE FUNCTIONS

G. R. MacLANE

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By

G. R. MacLANE

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RIEMANN SURFACES AND ASYMPTOTIC VALUES ASSOCIATED WITH REAL ENTIRE FUNCTIONS,

by Gerald R. MacLane, Associate Professor of Mathematics at the Rice Institute.

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I. PRELIMINARIES

1. As usual, numbers in square brackets refer to the references at the end of the pamphlet.

Sections, equations, and theorems are numbered consecutively throughout each chapter. References such as §13, (13), and Theorem 13 refer to section 13, equation (13), and Theorem 13, respectively, in the chapter in which the reference occurs. References to a different chapter are preceded by a Roman numeral specifying the chapter.

2. The purpose of this pamphlet is to give a detailed development of the results outlined in [13]; this is done in III. The results are summarized in III Theorems 3 (p. 63) and 4 (p. 64). Theorem 3 exposes a one-one correspondence between all real entire functions, $f'(z)$, ($f'(0)=1$), of finite order and with an infinite number of zeros, all of which are positive, and a class of Riemann surfaces (which is specified in detail in III, §§17–18); each surface is simply connected, parabolic, and has an infinite number of algebraic branch points and a finite number of logarithmic branch points.

Specifically, the correspondence is between $f'(z)$ and the Riemann surface for the inverse of the entire function $f(z)$. The reason for focussing attention on $f'(z)$ rather than $f(z)$ is that various questions of convergence in the proof of Theorem 3 depend heavily on the symmetry expressed by the fact that $f'(z)$ has only real positive zeros. In general the zeros of $f(z)$ have no such simple property.

Theorem 3 constitutes a very special extension of the results of Nevanlinna [16] and Elfving [5] on Riemann surfaces with a finite number of algebraic and logarithmic branch points. The complications arising from an infinite number of algebraic branch points are offset by the symmetry imposed.

III Theorem 4 is concerned with the asymptotic behavior of the functions $f(z)$ and $f'(z)$. The proof of Theorem 4 depends on Theorem 3; that is, both the specific form of $f'(z)$ as an infinite product and the structure of the associated Riemann surface is involved in the proof. Essentially, Theorem 4 describes the sectors (sometimes crooked) in the z-plane corresponding to the asymptotic values of $f'(z)$ and $f(z)$. These results involve no restrictions on the regularity of the sequence of zeros of $f'(z)$, which is in contrast to results such as those of Lindelöf [17, pp. 217–221].

3. The basic tool in the proof of III Theorem 3 is Carathéodory's theory of kernels [3, 4, 1, 2, 11, 6, 26]. Carathéodory's paper [3] contains all the essential ideas, but the formulation is not general enough for our purposes. The most complete treatment is that of Volkovyskiĭ [26]. However, Volkovyskiĭ's definition of kernel seems unnatural for the following reason: "exceptional points" should not be included in a kernel unless they are (in the appropriate sense, see II §25) included in all but a finite number of the Riemann surfaces in the sequence. Strangely enough, the non-uniqueness of kernels seems to have passed unnoticed (see II §21).

In II we give a reasonably complete development of the theory of kernels, culminating in II Theorem 11 (p. 33), which is the basic tool in proving III Theorem 3. We show by example that none of the hypotheses of II Theorem 11 can be dropped.

4. The method of proving convergence in III §§20–21 is essentially that used by Polya [20] and Lindwart and Polya [10] in considering the limiting functions of a sequence of polynomials with real zeros. For a general summary see Obrechkoff [18].

The rest of this chapter is devoted to definitions, theorems, and lemmas which are essential in II and III.

5. Notation and **terminology.** We shall use capital **script** letters to denote Riemann surfaces; in particular, $\mathcal{Z}$ and $\mathcal{W}$ are used to denote the z and w **spheres** respectively. By **z-plane** we mean the finite plane etc.

The symbols A', A^-, and A° are used to denote the **complement, closure,** and **interior,** respectively, of the set A as a subset of some space.

By a **Jordan curve** on a Riemann surface $\mathcal{J}$ we mean any simple closed curve on $\mathcal{J}$. The **genus** of $\mathcal{J}$ is the maximum number of disjoint Jordan curves which may be removed from $\mathcal{J}$ without disconnecting $\mathcal{J}$.

If $\mathcal{J}$ is a Riemann surface given as a covering of a Riemann surface $\mathcal{J}_0$, then we denote by $\mathcal{J}^\delta$ the unbranched covering of $\mathcal{J}_0$

obtained by deleting the branch points of $\mathcal{J}$ relative to $\mathcal{J}_0$. **Branch point** will always mean algebraic branch point; **logarithmic branch points** will be referred to by full title. The **order** of a branch point is the multiplicity of the corresponding zero of the appropriate derivative.

We shall say that the complex valued function $f(P)$ is **meromorphic** on $\mathcal{J}$ if and only if $f(P)$ is defined, uniform, and regular except for isolated poles on $\mathcal{J}$, that is, f is uniform and of rational character on $\mathcal{J}$. We shall say that f is **holomorphic** on $\mathcal{J}$ if and only if f is meromorphic on $\mathcal{J}$ and has no poles. A function f, meromorphic on $\mathcal{J}$, is **schlicht** if and only if $P_1, P_2 \in \mathcal{J}$, $P_1 \neq P_2$ implies $f(P_1) \neq f(P_2)$.

Let $w = f_n(P)$, $n \geqq 1$, be a sequence of meromorphic functions on $\mathcal{J}$. We shall say that $f_n(P) \to f(P)$ **subuniformly** on $\mathcal{J}$ if and only if $f_n(P) \to f(P)$ uniformly (in the spherical metric of $\mathcal{W}$) on any compact subset of $\mathcal{J}$. If $f_n \to f$ subuniformly on $\mathcal{J}$ then f is either $\equiv \infty$ or meromorphic on $\mathcal{J}$. The holomorphic functions $f_n(P)$ are **subuniformly bounded** on $\mathcal{J}$ if and only if they are uniformly bounded on any compact subset of $\mathcal{J}$. A family of meromorphic functions f_n is **normal** on $\mathcal{J}$ if and only if every subsequence of the functions contains a subsequence which converges subuniformly on $\mathcal{J}$.

6. We collect here several well known **distortion theorems,** in a form convenient for our purposes, to which we shall refer repeatedly. References for Theorems 1 and 2 will be found in [21, Abschn. IV] for example. The obvious changes of variable give Theorems 1a and 2a. For Theorem 3 see [7, p. 175].

THEOREM 1. (Koebe) *Let* $w = f(z) = a_0 + a_1 z + \cdots$ *be holomorphic and schlicht in* $|z| < R$ *and map* $|z| < R$ *onto a domain* D_w. *Then* D_w *contains the disc*

$$|w - a_0| < \tfrac{1}{4}|a_1| R.$$

Also,

$$\frac{|a_1 z|}{(1 + |z|/R)^2} \leqq |f(z) - a_0| \leqq \frac{|a_1 z|}{(1 - |z|/R)^2},$$

and

$$|a_1|\frac{1-|z|/R}{(1+|z|/R)^3}\leqq|f'(z)|\leqq|a_1|\frac{1+|z|/R}{(1-|z|/R)^3}.$$

THEOREM 1a. *Let* $w=f(z)=a_0+a_1/z+\cdots$ *be holomorphic and schlicht in* $|z|>R$ *and map* $|z|>R$ *onto a domain* D_w *of the* w*-plane. Then* D_w *contains the disc*

$$|w-a_0|<|a_1|/4R.$$

THEOREM 2. (Bieberbach) *Let* $w=f(z)=a_1z+a_0+\cdots$ *be schlicht in* $|z|>R$ *and map* $|z|>R$ *onto* $D_w\subset\mathcal{W}$. *Then* D_w *contains the disc* $|w-a_0|>2|a_1|R$.

THEOREM 2a. *Let* $w=f(z)=a_{-1}z^{-1}+a_0+\cdots$ *be schlicht in* $|z|<R$ *and map* $|z|<R$ *onto* $D_w\subset\mathcal{W}$. *Then* D_w *contains the disc* $|w-a_0|>2|a_{-1}|/R$.

THEOREM 3. *Let* D *be a domain in the* z*-plane, containing* $z=a$. *Let* $w=f(z)$ *be holomorphic and schlicht in* D, $f(a)=b$. *If* B *is a compact subset of* D, *then there exist constants* M *and* k, *depending only on* D, B, *and* a, *such that*

$$|f(z)-b|\leqq M|f'(a)|,\qquad z\in B,$$

$$\frac{1}{k}|f'(a)|\leqq|f'(z)|\leqq k|f'(a)|,\qquad z\in B.$$

The first inequality remains true for D *a domain in* $\mathcal{Z}$.

7. The following unicity theorem is easily proved by using the conformal map of the universal covering surface of $\mathcal{J}$ onto $|z|<R\leqq\infty$. A similar theorem of Rado [23] does not assume $f'(0)=1$; we make this assumption in order to include the punched sphere in the theorem.

THEOREM 4. *Let* $\mathcal{J}$ *be any Riemann surface which is (topologically) not the sphere. Let* P_0 *be a fixed point of* $\mathcal{J}$ *and* t *a local parameter at* P_0: $t=0$. *If the conformal map* $f\colon\mathcal{J}\to\mathcal{J}$ *is one-one onto a subsurface* $\mathcal{J}_1\subset\mathcal{J}$, *and if* $P_0\to P_0$ *and*

$$\left.\frac{d}{dt}f(t)\right|_{t=0}=1,$$

then $\mathcal{J}_1=\mathcal{J}$ *and* f *is the identity.*

8. The following three theorems of Lindelöf will be useful in Chapter III.

THEOREM 5. [17, p. 65]. *Let $f(z)$ be holomorphic and bounded in the sector $|\arg z| < \beta$. Let L be a continuous curve lying in $|\arg z| < \beta$ and running to ∞. Let*

$$\lim_{z\to\infty,\, z\in L} f(z) = a.$$

Then, for $\epsilon > 0$, $f(z) \to a$ as $z \to \infty$, uniformly in $|\arg z| \leqq \beta - \epsilon$.

THEOREM 6. [17, pp. 61–62]. *Let L_1 and L_2 be two simple continuous disjoint curves in the z-plane, each running from 0 to ∞. Let one of the domains bounded by $L_1 \cup L_2$ be D. If $f(z)$ is holomorphic and bounded in $D \cup L_1 \cup L_2$ and*

$$\lim_{z\to\infty} f(z) = a_n, \qquad z \in L_n,\ n = 1, 2$$

then $a_1 = a_2$ and $f(z) \to a_1$ uniformly as $z \to \infty$ in D.

THEOREM 7 [24, pp. 177–8]. *Let L_1, L_2, and D be as in Theorem 6. Let D be contained in the sector $|\arg z| < \beta$. Let $f(z)$ be holomorphic in $D \cup L_1 \cup L_2$ and satisfy*

$$|f(z)| < Ae^{|z|^\alpha}, \qquad A, \alpha \text{ constants},$$

in D and $|f(z)| < B$ on $L_1 \cup L_2$. If $\alpha\beta < \pi/2$ then $|f(z)| < B$ in D.

9. To investigate the asymptotic behavior of real entire functions we will make use of Iversen's theorem. Let $f(z)$ be meromorphic and let a be a constant. Let the family of (connected) domains D_ρ, $\rho > 0$ be characterized by:

1° D_ρ is bounded by level curves on which

$$|f(z) - a| = \rho, \qquad (|f(z)| = 1/\rho, \text{ if } a = \infty),$$

and

$$|f(z) - a| < \rho \text{ in } D_\rho, \qquad (|f(z)| > 1/\rho, \text{ if } a = \infty).$$

2° If $\rho_1 < \rho_2$ then $D_{\rho_1} \subset D_{\rho_2}$.

3° The domains D_ρ are all unbounded.

Such a family of domains defines a singularity of the Riemann surface $\mathcal{G}$ of the function inverse to $w = f(z)$. The set of all curves

Γ in the z-plane, running to ∞ and such that

$$\Gamma \cap \{|z| > M_\rho\} \subset D_\rho, \qquad \rho > 0,$$

for some function M_ρ form an **asymptotic tract** of $f(z)$ [25, p. 137]. As $z \to \infty$ on any such Γ, $f(z) \to a$, and a is the asymptotic value associated with the tract.

IVERSEN'S THEOREM. *There exists an asymptotic tract associated with each family of domains satisfying* 1°, 2°, *and* 3°. *The correspondence between the singularities of* $\mathcal{G}$ *and the asymptotic tracts of* $f(z)$ *is one-one.* [9], [25], [17, pp. 274–5].

10. In III we shall study certain entire functions $f(z)$ which have asymptotic values in various domains which are not of the form $a < \arg z < b$. To avoid cluttering up the development of III we give the appropriate definitions and results now.

A domain D of the z-plane will be called a **sector** if and only if D is simply connected and bounded by a simple continuous curve C with both ends at ∞; that is, on $\mathcal{Z}$, D is a Jordan domain with ∞ as a boundary point. A sector of the form $\alpha < \arg (z - z_0) < \beta$ will be called a **linear sector**; a sector which is not linear is **crooked.** If $\Gamma \subset D$ is a continuous curve: $z = z(t)$, $0 \leqq t < 1$, such that $z(t) \to \infty$ as $t \to 1$ we shall call Γ a **path** in D.

Let D be a sector in the z-plane and let $f(z)$ be meromorphic in D. Let $z = \Phi(W)$ map the linear sector $|\arg W| < \alpha$, α constant, $0 < \alpha \leqq \pi$, conformally onto D with the points at ∞ corresponding. Let $f(z) = f(\Phi(W)) = F(W)$. *We shall say that* $f(z)$ *has the asymptotic value* a *(finite or infinite) in* D *if and only if: for any* $\epsilon > 0$, $F(W) \to a$ *uniformly as* $W \to \infty$ *in* $|\arg W| \leqq \alpha - \epsilon$.

To justify this definition we note the following facts.

1) The choice of α is of no consequence; transformations of the form $W \to W^k$ allow us to alter α at will. *For the present we will assume that* $\alpha = \pi/2$.

2) The particular choice of the map Φ is immaterial so long as the points at ∞ correspond. This is clear if one considers the image of $|\arg W| \leqq \pi/2 - \epsilon$ under any linear transformation,

$$W \to AW + Bi, \qquad A \text{ and } B \text{ real}, A > 0,$$

mapping $\mathfrak{R}W > 0$ onto itself and preserving ∞.

3) We shall say that $f(z)$ has the asymptotic value *a uniformly* in D if and only if $f(z)\to a$ uniformly as $z\to\infty$ in D. By Carathéodory's theorem $z=\Phi(W)$ maps $\mathfrak{R}W\geqq 0$ continuously onto $D\cup C$ and hence

$$(1)\qquad \begin{cases} z = \Phi(W) \to \infty, & \textit{uniformly as } W \to \infty \textit{ in } \mathfrak{R}W \geqq 0,\\ W = \Phi^{-1}(z) \to \infty, & \textit{uniformly as } z \to \infty \textit{ in } D^{-}. \end{cases}$$

It follows that if $f(z)$ has the asymptotic value a uniformly in D then $f(z)$ has the asymptotic value a in D.

11. Instead of using the map Φ we may proceed as follows. Let z_0 be a fixed point of C; z_0 separates C into two infinite arcs, C_0 and C_1. Let $u(z)$ be the harmonic measure [17] of C_1 in D, that is, $u(z)$ is the unique function which is bounded and harmonic in D and assumes the boundary values 0 and 1 on C_0 and C_1 respectively. Denote by D_ϵ, $0<\epsilon<1/2$ the subdomain of D defined by $\epsilon<u(z)<1-\epsilon$. Then *$f(z)$ has the asymptotic value a in D if and only if: for all $\epsilon>0$, $f(z)\to a$ uniformly as $z\to\infty$ in D_ϵ.* That this property of $f(z)$ is equivalent to $F(W)\to a$ uniformly as $W\to\infty$ in $|\arg W|<\pi/2-\pi\epsilon$ is clear.

12. We now derive some useful lemmas which also show that asymptotic values associated with sectors, as defined in §10, behave in a reasonable fashion.

LEMMA 1. *Let D_1 and D_2 be two sectors in the z-plane such that*

$$D_1\cap\{|z|>M\} = D_2\cap\{|z|>M\}$$

for M sufficiently large. Let $f(z)$ be meromorphic in $D_1\cup D_2$. Then $f(z)$ has the asymptotic value a in D_1 if and only if $f(z)$ has the asymptotic value a in D_2.

Proof: If u_1 and u_2 are the two harmonic measures, as in §11, then, for M sufficiently large, u_1-u_2 is harmonic and bounded in $D_1\cap\{|z|>M\}$ and has the boundary value 0 except for points of $|z|=M$. Hence $u_1-u_2\to 0$ as $z\to\infty$ in D_1 or D_2. Thus

$$D_{1,\epsilon}\cap\{|z|>M_\epsilon\}\subset D_{2,\epsilon/2}\cap\{|z|>M_\epsilon\}$$

and if $f(z)$ has the asymptotic value a in D_2 then $f(z)$ has the asymptotic value a in D_1.

Thus alteration of any finite portion of D will not affect asymptotic values in D. The following lemma is also easy to prove.

LEMMA 2. *Let D_1 be a sector bounded by C_1. There exists a pair of curves C and C' running on either side of C_1 and bounding a neighborhood of C_1 such that: if D_2 is a sector whose boundary C_2, runs between C and C' and separates C and C', then any $f(z)$ meromorphic in $D_1 \cup D_2$ has the asymptotic value a in D_1 if and only if $f(z)$ has the asymptotic value a in D_2.*

LEMMA 3. *Let D and D' be two sectors in the z-plane, bounded by C and C', and such that D' is eventually contained in D, that is, there exists a constant M such that*

$$D' \cap \{|z| > M\} \subset D.$$

Let $u(z)$ be a harmonic measure associated with D as in §11. Let Γ be a path in D' such that for $z \in \Gamma$, $\delta \leqq u(z) \leqq 1-\delta$, where δ is a constant, $0 < \delta \leqq 1/2$.

Let $f(z)$ be meromorphic in $D \cup D'$. If $f(z)$ has the asymptotic value a in D then $f(z)$ has the asymptotic value a in D'.

The hypothesis that $u(z)$ is bounded away from 0 and 1 on some path $\Gamma \cap D'$ may be replaced by: D' contains a linear sector, $\alpha < \arg(z-z_0) < \beta$.

Proof: We note first that the hypothesis involving Γ is not extraneous. For example, let D be $\{x>0\}$ and D' be $\{y>0, 0<x<1\}$, where $z=x+iy$ and let $f(z)=e^{-z}$.

By virtue of Lemma 1 we may assume that $D' \subset D$, that C and C' have a common point z_0, and that Γ has one end at z_0. Here z_0 must be the point of C which determines $u(z)$ and the initial portion of Γ may be taken as a level curve of u, say $u=1/2$. D' may then be altered so as to include Γ and lie in D. Finally we may assume that Γ is simple.

Now z_0 splits C into C_0 and C_1 and $u(z)=0$ on C_0 and $u(z)=1$ on C_1. Also z_0 splits C' into the arcs C_0' and C_1', where C_1' separates C_1 and C_0' in D. Let $u_1(z)$ be the corresponding harmonic measure in D', that is, $u_1=1$ on C_1' and $u_1=0$ on C_0'. Let the subdomain of D' bounded by C_1' and Γ be D_1', and let the subdomain of D' bounded by C_0' and Γ be D_0'. We shall prove

Lemma 3 by showing that any domain D_ϵ' in which $\epsilon<u_1(z)<1-\epsilon$, $\epsilon>0$, is contained in some domain D_η defined by $\eta<u(z)<1-\eta$, $\eta>0$. This is easily accomplished by means of the maximum principle. For $z\in D_1'$, $\delta\leqq u$. Also, on Γ,

$$\frac{u-(1-\delta)}{\delta}\leqq 0\leqq u_1,$$

and on C_1',

$$\frac{u-(1-\delta)}{\delta}\leqq\frac{1-(1-\delta)}{\delta}=1=u_1.$$

Thus

$$\delta\leqq u(z)\leqq(1-\delta)+\delta u_1(z),\qquad z\in D_1'.$$

A similar argument proves that

$$\delta u_1(z)\leqq u(z)\leqq 1-\delta,\qquad z\in D_0'.$$

Since $0<u_1<1$ for $z\in D'$, we have

$$\delta u_1(z)\leqq u(z)\leqq(1-\delta)+\delta u_1(z),\qquad z\in D'.$$

Hence D_ϵ' is contained in D_η, where η is any positive number satisfying $\eta<\delta\epsilon$.

To prove the final statement in Lemma 3 we proceed as follows. By Lemma 1 we may assume that $z=0$ is a point common to C and C' and that D' contains the sector $|\arg z|<\alpha$. Here we use $z_0=0$ to determine C_1, C_0, and u, where C_1 is that half of C which is separated in D from $\arg z=0$ by $\arg z=\alpha$. By Carleman's principle of domain extension [17, p. 63] it is obvious that $u\leqq U$ in $|\arg z|<\alpha$, where U is harmonic and bounded in the z-plane slit along $\arg z=\alpha$ and such that for any fixed r, $0<r<\infty$, $U(re^{i(\alpha-0)})=1$ and $U(re^{i(\alpha+0)})=0$. That is, $U(z)=(\arg z-\alpha)/2\pi$, where $\alpha<\arg z<2\pi+\alpha$. In particular, $U=1-\alpha/2\pi$ on the positive real axis. Thus $u\leqq 1-\alpha/2\pi$ on $\arg z=0$. Similarly, $u\geqq\alpha/2\pi$ on $\arg z=0$. Thus we may take the ray $\arg z=0$ for Γ and $\delta=\alpha/2\pi$. This completes the proof of Lemma 3.

13. Instead of considering subsectors we may consider the union of two overlapping sectors. The basic result is Lemma 4.

As in the case of Lemma 3, Lemma 4 may be stated in terms of the behavior of the sectors in some neighborhood of ∞. To avoid uninteresting verbiage we give the statement in the simple form to which the more general case may be reduced by Lemma 1.

LEMMA 4. *Let D_1 and D_2 be two sectors in the z-plane, bounded by C_1 and C_2 respectively, and such that $C_1 \cap C_2$ consists of a single point z_0. Let the two arcs into which z_0 splits C_i be $C_{i,0}$ and $C_{i,1}$, $i=1, 2$. Let $C_{2,0} \subset D_1$ and $C_{2,1} \subset E_1$, where E_1 is the sector, not D_1, bounded by C_1. Then, with the proper labelling, $C_{1,0} \subset D_2$ and $C_{1,1} \subset E_2$ where E_2 is the sector, not D_2, bounded by C_2. Also, $D_1 \cap D_2 = I$ is a sector bounded by $C_{1,0}$, $C_{2,0}$, and z_0, and $D_1 \cup D_2 = D$ is a sector bounded by $C_{1,1}$, $C_{2,1}$ and z_0. Let u_1, u_2, and u be the harmonic measures of $C_{1,1}$, $C_{2,1}$, and $C_{1,1}$ in the domains D_1, D_2, and D respectively.*

Let there exist two paths, Γ_1 and Γ_2, in I and a constant δ, $0 < \delta \leqq 1/2$ such that

$$\delta \leqq u_1(z) \leqq 1 - \delta, \qquad z \in \Gamma_1, \tag{2a}$$

$$\delta \leqq u_2(z) \leqq 1 - \delta, \qquad z \in \Gamma_2. \tag{2b}$$

Let $f(z)$ be meromorphic in D and have the asymptotic values a and b in D_1 and D_2 respectively. Then $a = b$ and $f(z)$ has the asymptotic value a in D.

The hypothesis that there exist Γ_1, Γ_2, and δ satisfying (2) *may be replaced by: I contains a linear sector $\alpha < \arg (z - z_1) < \beta$.*

Remarks. We note first that the condition (2) is not superfluous. For example, let D_1 be $|\arg z| < \pi/2$, let D_2 be $\pi/2 < \arg (z-1) < \pi$, and let $f(z) = e^z$. In this example neither u_1 nor u_2 satisfy (2) on any paths $\Gamma \subset I$. If we change the example slightly we can show that it is essential to have both (2a) and (2b). Namely, let D_1 be $|\arg z| < \pi/2$, let D_2 be $\{|x| < 1, y > 0\}$, where $z = x + iy$, and let $f(z) = e^z/z$. Here we can satisfy (2b) but not (2a). In this example, although $f(z)$ has the asymptotic value ∞ in D, $a \neq b$.

Proof: By Lemma 1 we may alter D_1 and D_2 in the neighborhood of z_0 in such a way that Γ_1 and Γ_2 may be extended so as to start at z_0 while (2a) and (2b) are satisfied on the new Γ_1

and Γ_2 for some δ. This is most easily accomplished by taking six distinct radii of the disc $|z-z_0|<1$ for the arcs of C_1, C_2, Γ_1, and Γ_2 which lie in $|z-z_0|<1$.

Now $u_1 \geqq \delta$ on Γ_1 and $u_1=1>\delta$ on $C_{1,1}$. Since $C_{2,0}$ lies in the sector bounded by $C_{1,1}$ and Γ_1, $u_1+u_2=u_1\geqq\delta$ on $C_{2,0}$. Similarly, $u_1+u_2=u_2\geqq\delta$ on $C_{1,0}$. Hence $u_1+u_2\geqq\delta$ in I. Also $u_1+u_2=u_1\leqq 1$ on $C_{2,0}$ and $u_1+u_2=u_2\leqq 1$ on $C_{1,0}$ and therefore $u_1+u_2\leqq 1$ in I. Collecting:

$$\delta \leqq u_1(z) + u_2(z) \leqq 1, \qquad z \in I. \tag{3}$$

Since $u_1\geqq\delta$ on $C_{2,0}$, it follows that any level curve of u_1, $u_1=\epsilon$, $0<\epsilon<\delta$, is a path lying in I and issuing from z_0. For $\epsilon=\delta/2$ let this path be Γ. By (3) we have on Γ:

$$u_2 \leqq 1 - u_1 = 1 - \delta/2 \quad \text{and} \quad u_2 \geqq \delta - u_1 = \delta/2.$$

Thus we have reduced Lemma 4 to the special case where $\Gamma_1=\Gamma_2$, and, replacing $\delta/2$ by δ, we assume that (2a) and (2b) are satisfied on Γ. That $a=b$ is a consequence of the relations: $f(z)\to a$ on Γ and $f(z)\to b$ on Γ.

We shall complete the proof by showing that, for $\epsilon>0$, the subsector D_ϵ of D, defined by $\epsilon<u(z)<1-\epsilon$, is contained in $D_{1,\eta}\cup D_{2,\eta}$ for some $\eta>0$, where $D_{i,\eta}$ is the subsector of D_i characterized by $\eta<u_i(z)<1-\eta$, $i=1, 2$. For this purpose let H_i be the subsector of D_i bounded by $C_{i,1}$ and Γ, $i=1, 2$. Then $D=H_1\cup\Gamma\cup H_2$. Now, on $C_{1,1}$, $\delta u=\delta u_1<u_1$, while on Γ, $\delta u\leqq\delta \leqq u_1$. Thus $\delta u\leqq u_1$ in $H_1\cup\Gamma$. Also, by Carleman's principle of domain extension, $u_1<u$ in D_1, and hence in $H_1\cup\Gamma$. Thus $\delta u\leqq u_1\leqq u$ in $H_1\cup\Gamma$. It follows that

$$D_\epsilon \cap (H_1 \cup \Gamma) \subset D_{1,\eta}$$

where η is any positive number $<\epsilon\delta$. Similarly,

$$D_\epsilon \cap (H_2 \cup \Gamma) \subset D_{2,\eta}$$

for $\eta<\epsilon\delta$. Thus $D_\epsilon\subset D_{1,\eta}\cup D_{2,\eta}$ for $\eta<\epsilon\delta$.

Finally, to prove the last statement in Lemma 4, the argument at the end of §12 shows that the bisector of any linear sector contained in I will serve as a curve Γ on which both u_1 and u_2 are bounded away from 0 and 1. This completes the proof of Lemma 4.

14. Since we will be concerned with sectors in which both $f(z)$ and $f'(z)$ have asymptotic values, we shall need the following result.

LEMMA 5. *Let* $z=\psi(\zeta)$ *map a linear sector* D_ζ: $|\arg \zeta| < \alpha$, $0<\alpha\leqq\pi$, *onto a sector* D_z *in the z-plane with the points at* ∞ *corresponding. Then, if* $\zeta = re^{i\theta}$,

$$|\psi'(1)|\, r^{-1-\pi/\alpha} \cos \frac{\pi\theta}{2\alpha}\,(1+O(r^{-\pi/2\alpha}))$$

$$\leqq |\psi'(\zeta)| \leqq |\psi'(1)|\, r^{-1+\pi/\alpha} \sec^3 \frac{\pi\theta}{2\alpha}\,(1+O(r^{-\pi/2\alpha})), \quad r\to\infty.$$

Proof: This Lemma is a simple consequence of Theorem 1. Map D_ζ onto $|Z|<1$ by

$$Z = (1-\zeta^{\pi/2\alpha})/(1+\zeta^{\pi/2\alpha}).$$

Then $\zeta=\infty$ corresponds to $Z=-1$. It is easily verified that as $\zeta\to\infty$ in D_ζ,

$$|Z| = 1 - 2r^{-\pi/2\alpha} \cos \frac{\pi\theta}{2\alpha}\,(1+O(r^{-\pi/2\alpha}))$$

and

$$\left|\frac{dZ}{d\zeta}\right| = \frac{\pi}{\alpha}\, r^{-1-\pi/2\alpha}(1+O(r^{-\pi/2\alpha})).$$

Using these two results together with Theorem 1 we have Lemma 5.

LEMMA 6. *Let* $\psi(\zeta)$, D_ζ, *and* D_z *be as in Lemma 5. Let* $\epsilon>0$ *be fixed and let* $\{\zeta_n\}$ *and* $\{\zeta_n'\}$ *be two sequences of points in* $|\arg \zeta| \leqq \alpha-\epsilon$, *such that* $\zeta_n\to\infty$, $\zeta_n'\to\infty$ *as* $n\to\infty$. *Let* $z_n=\psi(\zeta_n)$ *and* $z_n'=\psi(\zeta_n')$. *If* $\zeta_n/\zeta_n'\to 1$ *as* $n\to\infty$, *then* $z_n/z_n'\to 1$ *as* $n\to\infty$.

Proof: The proof of Lemma 6 may be reduced to the case $\alpha=\pi/2$ by the transformation $\omega=\zeta^{\pi/2\alpha}$ so we shall assume that $\alpha=\pi/2$. Applying Theorem 1 to $\psi(\zeta)$ in the disc $|\zeta-\zeta_0| < \Re\zeta_0$ we obtain, since $\psi(0)$ is a boundary point of D_z,

$$|\psi(\zeta_0)-\psi(0)| \geqq \tfrac{1}{4}\,|\psi'(\zeta_0)|\,\Re\zeta_0.$$

Hence

$$\left|\frac{dz}{z}\right| \leqq 4\left|1-\frac{\psi(0)}{z}\right| \sec\theta\left|\frac{d\zeta}{\zeta}\right|, \qquad \zeta = re^{i\theta}.$$

Since $z\to\infty$ as $\zeta\to\infty$,

$$\left|\frac{dz}{z}\right| \leqq 8\sec\theta\left|\frac{d\zeta}{\zeta}\right|, \qquad r \geqq r_0,\ |\theta| < \frac{\pi}{2}. \tag{4}$$

On the segment, L_n, joining ζ_n and ζ_n',

$$|\zeta| \geqq \min\left(|\zeta_n|, |\zeta_n'|\right)\sin\epsilon.$$

Since $\zeta_n/\zeta_n'\to 1$ and $\zeta_n\to\infty$, we have

$$|\zeta| \geqq \tfrac{1}{2}|\zeta_n|\sin\epsilon \geqq r_0, \qquad \zeta \in L_n,\ n > n_0.$$

Integrating (4) over L_n,

$$\left|\log\frac{z_n'}{z_n}\right| \leqq \int_{L_n}\left|\frac{dz}{z}\right| \leqq \frac{16\csc^2\epsilon}{|\zeta_n|}\int_{L_n}|d\zeta| = 16\csc^2\epsilon\left|1-\frac{\zeta_n'}{\zeta_n}\right|.$$

By hypothesis, $\zeta_n'/\zeta_n\to 1$ and hence $\log(z_n'/z_n)\to 0$, and therefore $z_n'/z_n\to 1$.

II. CARATHÉODORY KERNELS

1. Kernels of Schlicht Domains. Let $\{D_n\}$ be a sequence of domains on the z-sphere Z such that $z=a\in D_n$, $n\geqq 1$. The kernel, K, of $\{D_n\}$ with respect to $z=a$ is defined as follows:

1°: if there is no fixed disc, $\{|z-a|<c\}$, contained in all D_n, then K consists of the single point $z=a$.

2°: if there exists a fixed disc $\{|z-a|<c\}$ contained in all D_n, then K is *the* maximal domain satisfying the two conditions:

$$z = a \in K, \tag{1}$$

(2) if $F\subset K$ and F is compact, then there exists $n_0=n_0(F)$ such that $F\subset D_n$ for $n>n_0$.

2. Existence of kernel. This definition requires justification in the following respect: we must show that, in case 2°, there exists a unique maximal domain K with the properties (1) and (2). The "maximal" here means of course that K is not a proper subdomain of a domain K_1 with the properties (1) and (2). An existence proof for K may be built along familiar lines as follows. Let $\{K_\lambda\}$, $\lambda\in\Lambda$, be the class of all domains satisfying (1) and (2). This class is non void, for $\{|z-a|<c\}$ is an element. Then the kernel is

$$K = \bigcup_{\lambda\in\Lambda} K_\lambda. \tag{3}$$

It is clear that K is open, connected, and satisfies (1). The maximal property is a trivial consequence of (3). To show that K satisfies (2), let F be any compact subset of K. Then F is covered by a finite number of the domains K_λ:

$$F \subset \bigcup_{n=1}^{N} K_{\lambda_n}.$$

By Lebesgue's lemma there exist compact sets F_n such that $F_n\subset K_{\lambda_n}$ and $F=\bigcup_{n=1}^{N}F_n$. By the definition of the K_λ, $F_n\subset D_k$ for $k>k_n$. Hence $F\subset D_k$ for $k>\max k_n$. This completes the existence proof for the kernel. The existence of K may also be proved more constructively; cf. §23.

3. In case 1° we shall call either the sequence $\{D_n\}$ or the kernel K *degenerate with respect to* $z=a$. The following theorem gives another characterization of K in the non-degenerate case, which provides a convenient method of actually determining the kernel. We denote the interior of any set S by S^o.

THEOREM 1. *Let* $\{D_n\}$ *be non-degenerate with respect to* $z=a$. *Set*

$$E_n = \bigcap_{k=n}^{\infty} D_k \tag{4}$$

and

$$L = \bigcup_{n=1}^{\infty} \overset{o}{E}_n = \lim_{n\to\infty} \overset{o}{E}_n. \tag{5}$$

Then $K=L_a$, *where* L_a *is that component of* L *which contains* $z=a$. Proof. Let F be any compact subset of K. Then there exists a compact set $F_1 \subset K$ such that $F \subset F_1^o$. Now $F_1 \subset E_n$ for $n > n_0$ and therefore $F \subset F_1^o \subset E_n^o \subset L$. Since F was an arbitrary compact subset of K it follows that $K \subset L$. Since $a \in K$ and K is connected, $K \subset L_a$. To obtain the reverse inclusion we shall show that L, and hence L_a, satisfies conditions (1) and (2). Since L is open, L_a is a domain and it will then follow from §2 that $L_a \subset K$. That L satisfies (1) is clear. As for (2), let F be any compact subset of L; then F is covered by a finite number of the sets E_n^o, and, since these sets are increasing, there exists m such that

$$F \subset \overset{o}{E}_m \subset E_m = \bigcap_{k=m}^{\infty} D_k,$$

and hence $F \subset D_n$ for $n \geq m$. q.e.d.

It is natural to attempt to replace L by

$$H = \liminf_{n\to\infty} D_n = \bigcup_{n=1}^{\infty} E_n$$

and consider that component, J, of H^o which contains a. We have of course $H^0 \supset L$ and $J \supset L_a = K$, but this inclusion may be proper, as shown by the following example. Let D_n be the disc $|z| < 1$ with the segment $\{\frac{1}{2} \leq |z| < 1,\ \arg z = 1/n\}$ removed, and let $a=0$. Then it is clear that $L=K$, which is the disc $|z|<1$ minus the slit $\{\frac{1}{2}\leqq|z|<1, \arg z=0\}$, while J is the complete disc $|z|<1$. We can conclude, however, that $J=K$ for monotonic sequences D_n.

COROLLARY 1. *If the sequence D_n is increasing, then K is non-degenerate and $K=\bigcup D_n$. If the sequence D_n is decreasing and non-degenerate with respect to a, then K is that component of* $(\lim D_n)^\circ$ *which contains a.*

This corollary follows immediately from Theorem 1.

4. We note that K depends on the base point a. The following two examples illuminate this dependence. The kernel of the sequence $D_n=\{|z|<1+1/n\}$ with respect to $z=0$ is $\{|z|<1\}$, while the kernel with respect to $z=1$ is $\{z=1\}$. Secondly, let D_n be the plane with all points of the imaginary axis outside $\{|z|<1/n\}$ removed. Then, with respect to $z=1$, K is the half-plane $\Re z>0$, while with respect to $z=-1$, K is the half-plane $\Re z<0$.

THEOREM 2. *Let $\{D_n\}$ be non-degenerate with respect to $z=a$. If there exists a fixed point $z=b$ such that $b\notin D_n$, $n\geqq n_0$, and if D_n is simply connected for $n\geqq n_0$, then K is simply connected.*

Proof. Let Γ be any Jordan curve in K and let Δ be that domain on $\mathcal{Z}$ determined by Γ which does not contain $z=b$. Then Γ, being compact, is contained in D_n for $n>n_1\geqq n_0$, and, since D_n is simply connected and does not contain b, $\Delta\cup\Gamma\subset D_n$. By the definition of the kernel, $\Delta\cup\Gamma\subset K$. Therefore K is simply connected. q.e.d.

5. The hypothesis that there exists $b\notin D_n$ is essential. It is not difficult to show that given any domain D and $a\in D$ that there exists a sequence of simply connected domains D_n on $\mathcal{Z}$ such that the kernel of $\{D_n\}$ with respect to a is D.

If the domains considered are all in the z-plane, then $b=\infty$ is always exterior and the existence of b may be dropped from the hypotheses of Theorem 2.

6. Definition of **convergence to kernel.** Let a, D_n, K be as heretofore. Let σ denote any subsequence, $\{\lambda_n\}$, of the positive integers and let K_σ denote the kernel of the subsequence $\{D_{\lambda_n}\}$ with respect to $z=a$. Obviously $K_\sigma\supset K$. We shall say

that $\{D_n\}$ *converges to its kernel* K *with respect to* $z=a$ *if and only if* $K_\sigma = K$ *for all* σ. In symbols we denote the convergence of D_n to K by $D_n \to K$.

The inclusion $K \subset K_\sigma$ is important for subsequent purposes.

7. We remark first that the convergence or non-convergence of $\{D_n\}$ depends upon the base point a, although it makes no difference so long as a is displaced in one fixed kernel of $\{D_n\}$. Let E_n be the z-plane cut along all of the imaginary axis except for the open interval between $-i/n$ and i/n. Set $D_{2n} = E_{2n}$ and $D_{2n+1} = E_{2n+1} \cap \{\Re z > -2\}$. Clearly $\{D_n\}$ converges to its kernel with respect to $z=1$, but not with respect to $z=-1$.

THEOREM 3. *Let* $\{D_n\}$ *be a monotonic sequence of domains and let* $a \in D_n$, $n \geqq 1$. *Then* $\{D_n\}$ *converges to its kernel with respect to* a.

Proof: Apparent by virtue of Corollary 1.

THEOREM 4. *Let* $D_n \to K$ *with respect to* $z=a$. *Let* p_n *be the connectivity of* D_n *and* p *the connectivity of* K. *Then*

$$p \leqq \liminf_{n\to\infty} p_n. \tag{6}$$

Proof: If $p=0$, i.e., $K=Z$, then K is compact and hence $D_n = K$ and $p_n = p = 0$ for $n > n_0$.

If $0 < p < \infty$, then there exists a domain Δ, bounded by p disjoint Jordan curves Γ_k, $1 \leqq k \leqq p$, $\Gamma = \bigcup \Gamma_k$, such that $\Delta \cup \Gamma \subset K$ and such that $E_k \not\subset K$, $1 \leqq k \leqq p$, where E_k is that Jordan domain bounded by Γ_k which does not meet Δ. We may also assume that $a \in \Delta$. Since $\Delta \cup \Gamma$ is compact, $\Delta \cup \Gamma \subset D_n$ for $n > n_0$. Now $E_k \not\subset D_n$ for $n > n_k$, for if $E_k \subset D_{\lambda_n}$, $n \geqq 1$, then the subsequence $\{D_{\lambda_n}\}$ would have a kernel with respect to a which would contain $\Delta \cup \Gamma \cup E_k$, which, with $E_k \not\subset K$, would contradict the convergence of $\{D_n\}$ to K. Hence for $n > \max_{0 \leqq k \leqq p} n_k$, D_n possesses one or more boundary continua in each of the p disjoint domains E_k and thus is of connectivity $\geqq p$. Therefore (6) is valid.

Finally, if $p = \infty$, we apply the above argument using domains Δ of arbitrary connectivity q. Then $\liminf p_n \geqq q$ for all integers q and therefore $\liminf p_n = \infty = p$. q.e.d.

8. That (6) is not necessarily valid without some hypothesis such as $D_n \to K$ is clear from §5. That (6) is the best possible result and that no inequality in the other direction is valid. even when the D_n are monotonic, may be seen from the following example. Let K be any given domain, not the whole sphere, of connectivity $p < \infty$. Let p_n be any sequence of integers such that $p \leqq p_n \leqq \infty$. There exists [28, p. 9] a sequence of domains E_n such that $E_n^- \subset E_{n+1}$, this inclusion being proper for $n \geqq 1$, such that $\bigcup E_n = K$, and such that $q_n \leqq p$, where q_n is the connectivity of E_n. Let D_n be E_n with exactly $p_n - q_n$ points removed from the fringe $E_n - E_{n-1}^-$; when $p_n = \infty$ this is to be interpreted as: $E_n - D_n$ is a countable closed subset of $E_n - E_{n-1}^-$. Then clearly the connectivity of D_n is p_n. Also $D_n \subset E_n \subset D_{n+1}$, and hence D_n is an increasing sequence with kernel $\bigcup D_n = \bigcup E_n = K$.

9. Carathéodory's Theorem. The following theorem is essentially that of Carathéodory [3, 4] who assumes that the domains D_n are all the unit circle and that the domains E_n are uniformly bounded. The more general statement of this theorem is due to Bieberbach [1, 2].

THEOREM 5. *Let* $\{D_n\}$ *be a sequence of domains in the z-sphere* Z. *Let* $z = 0 \in D_n$, $1 \leqq n$, *and let* $\{D_n\}$ *converge to its kernel* D *with respect to* $z = 0$. *Let* $\{E_n\}$ *be a sequence of domains in the w-plane such that* $w = b \in E_n$, $1 \leqq n$, *and let* E *be the kernel of* $\{E_n\}$ *with respect to* $w = b$. *Let*

$$w = f_n(z) \tag{7}$$

map D_n *schlichtly and directly conformally onto* E_n *and let*

$$\lim_{n\to\infty} f_n(0) = b \neq \infty, \qquad \lim f_n'(0) = c \neq 0,\ \infty. \tag{8}$$

If D *is non-degenerate then* 1) E *is non-degenerate and* 2) $E_n \to E$ *if and only if* $f_n(z)$ *approaches a holomorphic function* $f(z)$ *subuniformly in* D. *In the case of convergence* $w = f(z)$ *maps* D *schlichtly and conformally onto* E *with* $f(0) = b$, $f'(0) = c$.

If D *is degenerate then* E *is degenerate and* $E_n \to E = \{w = b\}$.

Remarks. For later purposes this formulation with (8) rather than the simpler conditions $f_n(0) = 0$, $f_n'(0) = 1$, is preferable.

In the degenerate case it is still true of course that $w = f_n(0) \to f(0) = b$ which maps $z = 0$ onto $w = b$.

It is important to note that the restriction of the E_n to be domains in the w-plane is essential, as shown by the following example. Let $D_n=\{|z|<1\}$, $E_n=\{|w-n/(n^2-1)|>1/(n^2-1)\}$, and $f_n(z)=z/(nz+1)$. Then $f_n(0)=0$, $f_n'(0)=1$, $D_n\to D=\{|z|<1\}$, which is non-degenerate, whereas $E_n\to E=\{w=0\}$ which is degenerate. This difficulty disappears if we remove the point $z=-1/n$ from D_n and the point $w=\infty$ from E_n. The new D_n, E_n, and f_n satisfy the hypotheses of Theorem 5, and D is now degenerate.

The crux of the matter is that the $f_n(z)$ need not form a normal family in D, whereas under the hypotheses of Theorem 5 $\{f_n\}$ is normal in D.

In some situations, e.g. if $\{D_n\}$ is an increasing sequence, $f_n(z)$ will not be defined in all of D. But the properties "$\{f_n(z)\}$ converges (does not converge) subuniformly in D" and "$\{f_n\}$ is normal (not normal) in D" are meaningful, as is apparent from the definition of kernel, for both properties are defined in terms of compact subsets of D.

10. The proof of Theorem 5 will be combined with the proof of the following extension of Theorem 5 in the non-degenerate case.

THEOREM 6. *Let* $\{D_n\}$, $\{E_n\}$, D, E, *and* $f_n(z)$ *satisfy the conditions of Theorem* 5 *through* (8), *except that the* E_n *are domains on the w-sphere. In addition let* D *be non-degenerate and let* $\{f_n\}$ *be normal in* D.

Then E *is non-degenerate and*

A). *If* $f_n(z)$ *approaches a meromorphic function* $f(z)$ *subuniformly in* D *then* $E_n\to E$ *and* $w=f(z)$ *maps* D *schlichtly and conformally onto* E *with* $f(0)=b$ *and* $f'(0)=c$.

B). *If* $E_n\to E$ *and* E *is not the complete sphere then* $f_n(z)\to f(z)$ *subuniformly in* D *and* $w=f(z)$ *maps* D *schlichtly and conformally onto* E *with* $f(0)=b$ *and* $f'(0)=c$.

The last conclusion is false if $E=\mathcal{W}$. For example, let D_n and D be $\mathcal{Z}$, E_n and E be $\mathcal{W}$, and $f_n(z)=z/(1+(-1)^n z)$.

11. Under the hypotheses of Theorem 5 it follows from (8) and I Theorem 3 that $\{f_n(z)\}$ is normal in D. Hence Theorem 5 follows from Theorem 6, except in the degenerate case. We dispose of this degenerate case of Theorem 5 first.

Let $\{D_n\}$ be degenerate with respect to $z=0$ and let $\{|z|<r_n\}$ be the largest disc with center at $z=0$ which is contained in D_n. Since $D_n \rightarrow \{z=0\}$ we have

$$\lim_{n\rightarrow\infty} r_n = 0. \tag{9}$$

Let $\{|w-f_n(0)|<\rho_n\}$ be the largest disc with center $f_n(0)$ which is contained in E_n. By I Theorem 1, $\rho_n \leqq r_n 4|f_n'(0)|$, and by (8) and (9), $\rho_n \rightarrow 0$ as $n\rightarrow\infty$. If E were non-degenerate then E would contain a disc $|w-b|<\rho$ and hence by (8) we would have $\rho_n \rightarrow \rho > 0$, which would be a contradiction. Hence E is degenerate.

12. We note next that the condition that $\{f_n(z)\}$ be normal in D is equivalent to: there exists r_0 such that $f_n(z)$ is holomorphic in $|z|<r_0$, that is, if $w=\infty\in E_n$, then the simple pole of $f_n(z)$ lies in $|z|\geqq r_0$. For, if $\{f_n(z)\}$ is normal the existence of r_0 follows from the first equation of (8) and the non-degeneracy of D. If there exists such an r_0 then $f_n(z)$ is holomorphic and schlicht in $|z|<r_0$ and by I Theorem 1 is subuniformly bounded, and therefore normal, in $|z|<r_0$. Also by I Theorem 1 the image of $|z|<r_0/2$ by $w=f_n(z)$ covers the disc $|w-f_n(0)|<r_0|f_n'(0)|/8$. It follows from (8) that this disc contains $|w-b|<r_0|c|/16$ for $n>n_0$ and therefore $f_n(z)$ is uniformly bounded away from b for $|z|\geqq r_0/2$. Hence $f_n(z)$ is normal in D.

13. THEOREM 7. *Let $\{D_n\}$ be a sequence of domains in $\mathfrak{Z}$; let $a\in D_n$ and let the kernel, D, of $\{D_n\}$ with respect to a be non-degenerate. Let $\{E_n\}$ be a sequence of domains in $\mathfrak{W}$ such that $w=b\in E_n$ and let E be the kernel of $\{E_n\}$ with respect to b.*

Let $w=f_n(z)$ map D_n schlichtly and conformally into *E_n and let*

$$\lim_{n\rightarrow\infty} f_n(a) = b \neq \infty, \qquad \lim_{n\rightarrow\infty} f_n'(a) = c \neq 0,\ \infty. \tag{10}$$

Then E is non-degenerate. If $f_n(z)\rightarrow f(z)$ subuniformly in D then $w=f(z)$ maps D schlichtly onto a subdomain of E.

Remarks. Note that we do not assume that $D_n\rightarrow D$ and that $w=f_n(z)$ need not map D_n *onto* E_n.

We shall prove Theorem 6 from Theorem 7 and then prove the latter.

14. To prove Theorem 6, A), let $f_n(z) \to f(z)$. By Theorem 7, $w = f(z)$ maps D onto $E^* \subset E$. The non-degeneracy of E follows from (8) and the non-degeneracy of D. Now let $\phi_n(w)$ be the function inverse to $f_n(z)$. It follows from (8) and I Theorem 1 that E contains a disc $|w-b| < \rho_0$ in which $\phi_n(w)$ has no poles. As noted in §12, the family $\{\phi_n(w)\}$ is normal in E. Let $\{\phi_{\lambda_n}(w)\}$ be any convergent subsequence of this family, $\phi_{\lambda_n}(w) \to \phi(w)$, and let the kernel of $\{E_{\lambda_n}\}$ be E_λ. By applying Theorem 7 to this subsequence of maps, $z = \phi(w)$ maps E_λ schlichtly into the kernel, D_λ, of $\{D_{\lambda_n}\}$. Since $D_n \to D$, $D_\lambda = D$. Also (cf. §6) $E_\lambda \supset E$, and hence $z = \phi(w)$ maps E into D.

The composite function $w_1 = f(\phi(w)) = \psi(w)$ maps E_λ into E^* with $\psi(b) = b$ and $\psi'(b) = 1$. Since $f_{\lambda_n}(\phi_{\lambda_n}(w))$ is defined and $= w$ on any compact subset of E_λ for n sufficiently large, it follows on letting $n \to \infty$ that $\psi(w) = w$. Hence $E_\lambda = E^*$. Since $E_\lambda \supset E \supset E^*$, we have $E^* = E = E_\lambda$. It follows from $E^* = E$ that $w = f(z)$ maps D *onto* E. From $E = E_\lambda$ we derive $E_n \to E$, for if E_μ is the kernel of any subsequence E_{μ_n} then the λ_n may be chosen as a subsequence of the μ_n, whence $E_\mu \subset E_\lambda = E$ and thus $E_\mu = E$.

15. To prove Theorem 6, B), assume that $E_n \to E$ where E is not the whole sphere. By the normality of $\{f_n\}$ there exists $\{\lambda_n\}$ such that $f_{\lambda_n}(z) \to f(z)$. Since $D_{\lambda_n} \to D$, it follows from §14 that $w = f(z)$ maps D onto E, the kernel of $\{E_{\lambda_n}\}$. Finally, $f_n \to f$, for otherwise by the normality of $\{f_n\}$ there would exist $\{\mu_n\}$ such that $f_{\mu_n} \to g \neq f$ and such that $w = g(z)$ would map D onto E with the same normalization: $g(a) = b$, $g'(a) = c$. By I Theorem 4 we would have the contradiction $g = f$. Therefore $f_n \to f$ subuniformly in D. This completes the proof of Theorem 6 from Theorem 7.

16. Proof of Theorem 7. That E is non-degenerate is apparent from (10), the non-degeneracy of D, and I Theorem 1.

Assume now that $f_n(z) \to f(z)$ subuniformly in D. By (10), $f(a) = b$ and $f'(a) = c \neq 0$; therefore $f(z)$ is not a constant and, being a limit of schlicht functions, is schlicht. We must show then that $w = f(z)$ maps D into E.

Since $f(z)$ is schlicht in D it has at most one pole in D, at $z = p \neq a$. If p exists remove this point from D; also delete ∞ if $\infty \in D$. Let D^* be the subdomain of D remaining after these possible deletions.

17. We prove first that if $w=f(z)$ maps D^* into E, then $f(z)$ also carries D into E. If the point $p\neq\infty$ exists then there exists a deleted neighborhood of ∞: $V^*=\{R<|w|<\infty\}\subset E$ which is the image of a deleted neighborhood $U^*\subset D^*$ of $z=p$. Since $f_n\to f$, each $f_n(z)$, for n sufficiently large, has a pole at $z=p_n\in D_n$, with $p_n\to p$. For n sufficiently large, $\{|z-p_n|\leqq r_0\}\subset D\cap D_n$, where r_0 is a fixed positive constant. Let

$$f_n(z)=\frac{A_n}{z-p_n}+B_n+\cdots,\qquad f(z)=\frac{A}{z-p}+B+\cdots.$$

Then $A_n\to A\neq 0$ and $B_n\to B$. By I Theorem 2a, E_n contains the disc $|w-B_n|>2|A_n|/r_0$. Hence E_n contains

$$|w-B|\geqq 4|A|/r_0,$$

and, a fortiori,

$$\left\{\infty\geqq|w|\geqq B+\frac{4|A|}{r_0}\right\}\subset E_n,\qquad n>n_1. \tag{11}$$

Since $\{R<|w|<\infty\}\subset E$,

$$\left\{2R\leqq|w|\leqq 2|B|+\frac{8|A|}{r_0}\right\}\subset E_n,\qquad n>n_2. \tag{12}$$

From (11) and (12) it follows that

$$\{2R\leqq|w|\leqq\infty\}\subset E_n,\qquad n>n_3,$$

and hence, by the definition of kernel,

$$\{2R\leqq|w|\leqq\infty\}\subset E,$$

which implies that $w=\infty\in E$. Hence $f(p)=\infty\in E$.

If p exists and $p=\infty$ the argument is essentially unchanged; we use I Theorem 2 rather than I Theorem 2a.

Finally, if $\infty\in D$, we may assume that $f(\infty)\neq\infty$, for the case $p=\infty$ has been disposed of. Again the argument is essentially the same and hinges on I Theorem 1a.

18. To prove now that $w=f(z)$ maps D^* into E, let z_0 be any point of D^* and let Γ be a polygonal arc in D^* joining $z=a$ to $z=z_0$. Then

$$\operatorname{dist}(\Gamma, D^{*\prime}) = \delta > 0.$$

Here we use the metric of the plane, not the sphere.

Let A denote the compact set

$$A = \left\{z : \operatorname{dist}(z, \Gamma) \leqq \frac{\delta}{2}\right\} \subset D^*$$

and B the open set

$$B = \left\{z : \operatorname{dist}(z, \Gamma) < \frac{2}{3}\delta\right\} \subset D^*.$$

Actually, $B^- \subset D^*$. Now $\infty \in B^-$, and $p_n \in B^-$ for infinitely many n would imply $p \in B^- \subset D^*$, which is incompatible with $p \notin D^*$. Hence $B^- \subset D_n^*$, (D_n less p_n and ∞) for $n > n_0$.

Let $f_n(\Gamma) = C_n$ and $f(\Gamma) = C$. Using (10) and applying I Theorem 3 to $A \subset B$,

$$|f_n(z)| \leqq M \text{ and } \frac{1}{k} \leqq |f_n'(z)| \leqq k, \qquad z \in A,\ n > n_1,$$

and these same estimates are valid for $f(z)$ and $f'(z)$. For $n > n_2(\epsilon)$, $|f_n(z) - f(z)| < \epsilon$ on A. The disc $|z - \zeta| < \delta/2$, $\zeta \in \Gamma$, is in A and is mapped by $f_n(z)$ or $f(z)$ onto a domain of the w-plane containing

$$|w - f_n(\zeta)| < \frac{\delta}{8k} \quad \text{or} \quad |w - f(\zeta)| < \frac{\delta}{8k}\cdot$$

Thus

$$\left\{w : \operatorname{dist}(C_n, w) < \frac{\delta}{8k}\right\} \subset E_n, \qquad n > n_1.$$

If $\epsilon < \delta/16k$, then

$$G = \left\{w : \operatorname{dist}(C, w) < \frac{\delta}{16k}\right\} \subset E_n, \qquad n > \max(n_1, n_2).$$

But G possesses the properties (except for maximality) of the kernel E, hence $G \subset E$ and a fortiori $w_0 = f(z_0) \in E$. Since z_0 was any point of D^*, $w=f(z)$ carries D^* into E. This completes the proof of Theorem 7 and hence the proof of Theorems 5 and 6.

KERNELS OF SEQUENCES OF RIEMANN SURFACES

19. To define a kernel of a sequence $\{\mathcal{J}_n\}$ of Riemann surfaces it is necessary to have some relations connecting the various $\mathcal{J}_n$; that is, the $\mathcal{J}_n$ may not be given abstractly with no specific connections. The natural set up here is to consider the $\mathcal{J}_n$ as given covering surfaces (in general both branched and bordered) of a fixed base surface $\mathcal{B}$. Since any Riemann surface is conformally equivalent to a covering of the sphere it is no real restriction to assume that $\mathcal{B}$ is the z-sphere which we denote by $\mathcal{Z}$.

20. Embedding. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are two Riemann surfaces over $\mathcal{Z}$ and $P_1 \in \mathcal{J}_1$ and $P_2 \in \mathcal{J}_2$ are two points over $z=a$, we shall say that $\mathcal{J}_1$ is $\mathcal{Z}$-embeddable in $\mathcal{J}_2$ with P_1 corresponding to P_2 if and only if there exists a homeomorphism T of $\mathcal{J}_1$ into $\mathcal{J}_2$ which preserves z and carries P_1 into P_2. Clearly if T exists it is unique, provided P_1 is not a branch point of $\mathcal{J}_1$ relative to $\mathcal{Z}$. Except for exceptional symmetrical situations, T is unique even when P_1 is a branch point. If T exists we shall denote this relation between $\mathcal{J}_1$ and $\mathcal{J}_2$ by

$$[\mathcal{J}_1; P_1] \subset_z [\mathcal{J}_2; P_2]. \tag{13}$$

The subscript z will be dropped when there is no danger of confusion. If T is onto we shall use $=_z$ (or $=$) in place of $\subset$. The symbol $[\mathcal{J}; P]$ will be used in general to denote a Riemann surface $\mathcal{J}$ with a specified point P. If P_1' is another point of $\mathcal{J}_1$ over a it is of course possible to have (13) valid and also

$$[\mathcal{J}_1; P_1'] \not\subset [\mathcal{J}_2; P_2].$$

A similar remark applies to alterations of P_2.

An obvious but useful criterion for embeddability is contained in the following theorem. The proof is clear and we omit it.

THEOREM 8. *Let $\mathcal{J}_1$, $\mathcal{J}_2$, P_1, and P_2 be as described and let P_1 be a non-branch point of $\mathcal{J}_1$ relative to $\mathcal{Z}$. Then $[\mathcal{J}_1; P_1] \subset [\mathcal{J}_2; P_2]$ if and only if the following two conditions hold:*

1° *Let Q_1 be any point of $\mathcal{J}_1$ and let Γ_1 be any curve in $\mathcal{J}_1$ joining*

P_1 to Q_1. Let Γ_1 contain no branch points (relative to Z) of $\mathcal{J}_1$ with the possible exception of Q_1, and let the equation of Γ_1 be $z=z(t)$, $0 \leqq t \leqq 1$, $z(0)=P_1$, $z(1)=Q_1$. Then a curve Γ_2 exists in $\mathcal{J}_2$ with the same equation $z=z(t)$, joining P_2 to $z(1)=Q_2 \in \mathcal{J}_2$, and containing no branch points of $\mathcal{J}_2$ with the possible exception of Q_2.

2° If Γ_1, Γ_2 and Γ_1', Γ_2' are two pairs of curves of the type described in 1°, then Γ_1, Γ_1' have the same end point $Q_1 \in \mathcal{J}_1$, if and only if Γ_2 and Γ_2' have the same end point $Q_2 \in \mathcal{J}_2$.

If 1° and 2° are satisfied, then $Q_1 \rightarrow Q_2$ gives the embedding T of $[\mathcal{J}_1; P_1]$ into $[\mathcal{J}_2; P_2]$.

Another obvious but useful result is the following:

THEOREM 9. *Let $\{\mathcal{E}_n; P_n\}$ be a sequence of Riemann surfaces over Z with P_n over $z=a$ not a branch point of $\mathcal{E}_n$. Let*

$$[\mathcal{E}_n; P_n] \subset [\mathcal{E}_{n+1}; P_{n+1}], \qquad n \geqq 1. \tag{14}$$

Then there exists a unique surface $[\mathcal{J}; P]$ over Z such that $[\mathcal{J}; P]$ contains a sequence of subsurfaces $[\mathcal{E}_n'; P]$ with the properties:

$$\mathcal{J} = \bigcup_{n=1}^{\infty} \mathcal{E}_n' \tag{15}$$

and

$$[\mathcal{E}_n; P_n] = [\mathcal{E}_n'; P]. \tag{16}$$

That is, to put it briefly, an increasing sequence of Riemann surfaces determines a unique Riemann surface by exhaustion. The proof of Theorem 9 is trivial. Since (16) is an equivalence relation we may identify $\mathcal{E}_n$ and $\mathcal{E}_n'$. Similarly the stated unicity of $[\mathcal{J}; P]$ is modulo this equivalence relation.

21. Let $\{\mathcal{J}_n\}$ be a sequence of Riemann surfaces over Z satisfying

(17) *$\mathcal{J}_n$ contains a specified point P_n over $z = a$, and a schlicht disc over $|z-a| < r_0$ about P_n.*

In defining kernel we encounter two complications here which did not crop up in the case of schlicht domains. The first is owing to the possible behavior of branch points. For example, let $\mathcal{J}_n$ be the two sheeted Riemann surface for $(z-1-1/n)^{1/2}$, which has two first order branch points, over $z=1+1/n$ and over $z=\infty$. Let $a=0$. Then clearly the Riemann surface $\mathcal{J}$ which ought to be called the kernel is the Riemann surface for $(z-1)^{1/2}$. But no compact subset of $\mathcal{J}$ which contains the branch point over $z=1$ is Z embeddable in any $\mathcal{J}_n$. This type of difficulty will be overcome by temporarily removing all branch points.

The second complication is that there is no unique kernel in general. For example, let $\mathcal{J}_{2n}$ be the schlicht disc over $|z|<1$ and let $\mathcal{J}_{2n+1}$ be the two sheeted unbordered covering of $|z|<1$ with a single branch point over $z=1/2$ .Let $a=0$. Then the disc $|z|<1$ cut along any Jordan arc Γ joining $z=1/2$ to $|z|=1$ and not passing through $z=0$ is a maximal Riemann surface embeddable in all $\mathcal{J}_n$; nor does this exhaust the possibilities, for we may use other types of continua to join $z=1/2$ and $|z|=1$ which still give maximal surfaces with the proper embeddability property. Clearly all of these maximal surfaces have equal claims to the title of kernel.

22. Definition of kernel for unbranched surfaces. Let $\{\mathcal{J}_n; P_n\}$, a, and r_0 satisfy (17). In addition let all $\mathcal{J}_n$ be unbranched over Z. Let $\mathcal{J}$ be a Riemann surface over Z and let P be a point of $\mathcal{J}$ over a. Then $[\mathcal{J}; P]$ is **a** kernel of $\{\mathcal{J}_n; P_n\}$ if and only if $[\mathcal{J}; P]$ is a maximal Riemann surface with the property:

1° *If* $\mathcal{A}$ *is any subsurface of* $\mathcal{J}$ *whose closure* $\mathcal{A}^-$ *in* $\mathcal{J}$ *is compact and such that* $P\in\mathcal{A}$, *then*

$$[\mathcal{A}; P]\subset[\mathcal{J}_n; P_n] \qquad \text{for } n>n_0(\mathcal{A}).$$

The maximality here of course means:

2° *If* $[\mathcal{J}'; P']$ *has the property* 1° *and if* $[\mathcal{J}; P]\subset[\mathcal{J}'; P']$ *then* $[\mathcal{J}; P]=[\mathcal{J}'; P']$.

We note that although any kernel $\mathcal{J}$ contains a disc about P, it need not contain a disc of radius as large as r_0; this is clear from the last example of §21. Also all kernels $\mathcal{J}$ are unbranched over Z.

23. Existence of kernels. Although there is no unicity in general, any sequence $\{\mathcal{J}_n; P_n\}$ satisfying (17) always possesses at least one kernel $\mathcal{J}$ as we now prove.

Consider all open discs of Z with rational centers and rational radii; since we are working on Z this means that with each "rational" circumference we associate two open discs. Let $\{\Delta_n\}$ be a sequential arrangement of these discs in which each disc occurs infinitely often. We shall assume that Δ_1 has center $z=a$ and radius $\leqq r_0$.

Any unbranched Riemann surface over Z may be built up with a countable number of these discs with suitable identifications.

Let $\mathcal{E}_1$ be the disc Δ_1, and let P denote the center of $\mathcal{E}_1$. Then $\mathcal{E}_1$ has the property 1°. If $\mathcal{E}_1 \cup \Delta_2$ is connected and has the property 1°, set $\mathcal{E}_2 = \mathcal{E}_1 \cup \Delta_2$; otherwise set $\mathcal{E}_2 = \mathcal{E}_1$. Proceeding in this fashion we obtain a sequence $\{\mathcal{E}_n\}$ of Riemann surfaces with the properties:

(α) $[\mathcal{E}_n; P] \subset [\mathcal{E}_{n+1}; P]$,

(β) $[\mathcal{E}_n; P]$ *satisfies* 1°,

(γ) $\mathcal{E}_{n+1}$ *consists of* $\mathcal{E}_n$ *and a finite* (*possibly* 0) *number of copies of* Δ_{n+1} *with appropriate identifications between subsets of* $\mathcal{E}_n$ *and subsets of the added discs, and*

(δ) *It is impossible to add another copy of* Δ_{n+1} *to* $\mathcal{E}_{n+1}$ *so that* 1° *is still satisfied.*

We prove by induction that this construction is possible and unique. Certainly $\mathcal{E}_1$ and $\mathcal{E}_2$, satisfy these four conditions. Assume that $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n$ have been obtained. It follows from the definition of $\mathcal{E}_1$ and (γ) that $\mathcal{E}_n$ is built of a finite number of discs Δ. If Δ_{n+1} has no points in common with the projection of $\mathcal{E}_n$ into Z, set $\mathcal{E}_{n+1} = \mathcal{E}_n$. Otherwise there is a finite set of subdomains, D_k, $1 \leqq k \leqq N$, of $\mathcal{E}_n$ which lie over Δ_{n+1}. If a given D is not schlicht over Δ_{n+1} it is impossible (cf. Theorem 8) to attach a copy of Δ_{n+1} to D. Let the D's which are schlicht over Δ_{n+1} be $D_1, D_2, \cdots, D_\mu$. If it is possible to attach a copy, Δ^1_{n+1}, of Δ_{n+1} to D_1, (by identification of D_1 with the projection of D_1 into Δ_{n+1}), so that $\mathcal{E}_n \cup \Delta^1_{n-1}$ satisfies 1°, then the attachment or non-attachment of Δ^1_{n+1} to D_ν, $2 \leqq \nu \leqq \mu$ is uniquely determined by this same condition 1°. The remaining D_ν are treated in a similar fashion. This process terminates in a finite number of steps and yields the desired $\mathcal{E}_{n+1}$. That $\mathcal{E}_{n+1}$ satisfies (α), (β), (γ), and (δ) is clear from the construction.

By Theorem 9 $\{\mathcal{E}_n\}$ determines a unique surface $\mathcal{J} = \bigcup \mathcal{E}_n$.

That $\mathcal{J}$ has the property 1° is easily proved, for if $\mathcal{A}$ is any subsurface of $\mathcal{J}$ whose closure $\mathcal{A}^-$ in $\mathcal{J}$ is compact, then $\mathcal{A}^-$ is covered by a finite number of the $\mathcal{E}_n$ and hence by a single $\mathcal{E}_n$. The conclusion then follows from (β).

Finally, $\mathcal{J}$ is maximal, for if not we would have a surface $[\mathcal{J}; P']$ satisfying 1°, and the proper inclusion $[\mathcal{J}; P] \subset [\mathcal{J}'; P']$. Now $\mathcal{J}'$ would consist of exactly the same discs Δ and identifications as $\mathcal{J}$ with some additional Δ's added. At least one of these additional discs, say Δ^*, would have points in common with $\mathcal{J}$, for $\mathcal{J}'$ is connected, and hence with an $\mathcal{E}_{n_0}$. Now Δ^* occurs infinitely many times in $\{\Delta_n\}$ and therefore occurs with an index $N > n_0$. Clearly by (δ) Δ^* is either part of $\mathcal{E}_{N-1}$ or was added in constructing $\mathcal{E}_N$, and hence Δ^* cannot be "additional." Thus $\mathcal{J}$ is maximal.

24. This process for obtaining a kernel leads to a unique kernel once the sequence $\{\Delta_n\}$ is assigned. More generally we may replace $\mathcal{E}_1 = \Delta_1$ by any surface $[\mathcal{E}_1; P]$ which has the property 1° and then proceed as before. Hence *if* $[\mathcal{E}_1; P]$ *satisfies* 1°, *then* $\{\mathcal{J}_n; P_n\}$ *has a kernel containing* $[\mathcal{E}_1; P]$.

The existence of a kernel may also be proved by Zorn's lemma.

25. Definition of kernel in general case. Let $\}\mathcal{J}_n; P_n\}$, a, and r_0 satisfy (17). Let $\mathcal{J}_n^\delta$ be the unbranched subsurface of $\mathcal{J}_n$ obtained by deleting the branch points of $\mathcal{J}_n$. Let $[\mathcal{J}^\delta; P]$ be any kernel of $\{\mathcal{J}_n^\delta; P_n\}$. The kernel, $\mathcal{J}$, of $\{\mathcal{J}_n\}$ corresponding to $\mathcal{J}^\delta$ is obtained by adding the proper branch points to $\mathcal{J}^\delta$. Precisely, for each $b \in Z$ and $r > 0$, consider each domain $\mathcal{D}^\delta$ of $\mathcal{J}^\delta$ which is an unbordered covering of $\{0 < |z-b| < r\}$, $(\{r < |z| < \infty\}$, if $b = \infty)$, of multiplicity m, $1 < m < \infty$. Let $\mathcal{A}_\rho^\delta$ be that part of $\mathcal{D}^\delta$ over $\{0 < \rho < |z-b| < r/2\}$. Since $\mathcal{A}_\rho^{\delta-}$ is compact, $\mathcal{A}_\rho^\delta \subset \mathcal{J}_n^\delta \subset \mathcal{J}_n$, for n large enough. The actual inclusion here is determined by $\mathcal{A}_\rho^\delta$ and a fixed curve on $\mathcal{J}^\delta$ joining P to $\mathcal{A}_\rho^\delta$. Now $\mathcal{J}_n$ contains one or more domains lying over $\{|z-b| < r/2\}$; let $\mathcal{B}_n$ be that in which $\mathcal{A}_\rho^\delta$ is embedded. We add a single point over $z = b$ to $\mathcal{D}^\delta \subset \mathcal{J}^\delta$, creating a branch point of order $m-1$, if and only if, for $n > n_0$, each $\mathcal{B}_n$ is a simply connected relatively unbordered m-sheeted covering of $\{|z-b| < r/2\}$, (with total branch order $m-1$). There are at most a countable number of domains $\mathcal{D}^\delta$ to consider, and having added a branch point to each one satisfying the specified condition, we obtain a unique $[\mathcal{J}; P]$, a kernel of $\{\mathcal{J}_n; P_n\}$. There is clearly a one-one correspondence between the kernels $\mathcal{J}^\delta$ and the kernels $\mathcal{J}$. We shall call $\mathcal{J}^\delta$ the *deleted kernel* corresponding to $\mathcal{J}$.

In defining kernels we have assumed that no P_n was a branch point. It is possible to extend the definition to include the case where all P_n are branch points of the same order.

26. Definition of convergence to kernel. *The sequence* $\{\mathcal{J}_n; P_n\}$ *converges to its kernel* $[\mathcal{J}; P]$ *if and only if*

(18) $\{\mathcal{J}_n; P_n\}$ *has a unique kernel* $[\mathcal{J}; P]$,

and

(19) $\begin{cases} \textit{every subsequence } \{\mathcal{J}_{\lambda_n}; P_{\lambda_n}\} \textit{ has the same} \\ \textit{unique kernel } [\mathcal{J}; P]. \end{cases}$

The statement, "$\{\mathcal{J}_n; P_n\}$ converges to its kernel $[\mathcal{J}; P]$," will be abbreviated to "$[\mathcal{J}_n; P_n] \to [\mathcal{J}; P]$" or simply "$\mathcal{J}_n \to \mathcal{J}$."

It should be noted that the definitions of kernel and convergence jibe with the definitions in the plane case when all the $\mathcal{J}_n$ are schlicht over $\mathcal{Z}$.

27. Structure of kernels. Various hypotheses on the $\mathcal{J}_n$ entail various properties of the kernels $\mathcal{J}$; but the situation is not as simple as in the plane case. The following facts are of interest.

A). If $\{\mathcal{J}_n\}$ is increasing (in the sense of Theorem 9) then

$$\mathcal{J}_n \to \mathcal{J} = \bigcup \mathcal{J}_n.$$

B). $\mathcal{J}_n^\delta \to \mathcal{J}^\delta$ does *not* imply $\mathcal{J}_n \to \mathcal{J}$.

C). If all $\mathcal{J}_n$ are schlichtartig then any kernel $\mathcal{J}$ is schlichtartig.

D). If $\mathcal{J}_n$ and $\mathcal{J}$ are of genus p_n and p respectively, where $\mathcal{J}$ is any kernel of $\{\mathcal{J}_n\}$, then

$$p \leqq \liminf_{n \to \infty} p_n.$$

E). $\mathcal{J}_n$ simply connected, $\mathcal{J}_n \to \mathcal{J}$ does *not* imply that $\mathcal{J}$ is simply connected.

The proof of A) is clear from the definitions of kernel and convergence. A simple example to illustrate B) is this: let $\mathcal{E}$

denote the two sheeted unbordered covering of $|z| < 1$ with a single branch point over $z = 1/2$, and let $a = 0$. Let $\mathcal{J}_n = \mathcal{E}$ or $\mathcal{E}^\delta$ according as n is odd or even.

To prove C), let $[\mathcal{J}; P]$ be any kernel of $\{\mathcal{J}_n; P_n\}$. If $\mathcal{J}$ is not schlichtartig then there exists a retrosection of $\mathcal{J}$ consisting of two Jordan curves Γ_1 and Γ_2 on $\mathcal{J}^\delta$ which are disjoint except for a single point Q where Γ_1 and Γ_2 cross. Then $\Gamma_1 \cup \Gamma_2$ is contained in a domain $\mathcal{A} \subset \mathcal{J}^\delta$ such that $[\mathcal{A}; P] \subset [\mathcal{J}_n; P_n]$, for n sufficiently large, since $\mathcal{J}$ is a kernel. Hence $\mathcal{J}_n$ contains a retrosection and is therefore not schlichtartig for $n > n_0$.

The proof of D) is analogous. Namely any set of disjoint Jordan curves on $\mathcal{J}$ which do not disconnect $\mathcal{J}$ leads to a similar set on each $\mathcal{J}_n$ for $n > n_0$.

Finally, as to E), let $\mathcal{J}_n$ be the two sheeted Riemann surface of $(z-n)^{1/2}$ with the point over $z = 1$ removed from the "lower sheet" and let P_n lie over $z = 0$ in the "lower sheet." It is clear that $\mathcal{J}_n$ is simply connected and that $\mathcal{J}_n \to \mathcal{J}$, where $\mathcal{J}$ is $\mathcal{Z}$ with the two points 1 and ∞ deleted.

28. Before considering the extension of Theorem 6 there is one point that requires discussion; namely the **definition of subuniform convergence on a kernel.** Let $[\mathcal{J}; P]$ be a kernel of $\{\mathcal{J}_n; P_n\}$ and let $\mathcal{J}^\delta$ be the deleted kernel corresponding to $\mathcal{J}$. Let $f_n(z)$ be meromorphic on $\mathcal{J}_n$. We shall say that $f_n(z) \to f(z)$ *subuniformly on* $\mathcal{J}^\delta$ *if and only if* $f_n(z) \to f(z)$ *uniformly on any subdomain* $\mathcal{A} \subset \mathcal{J}^\delta$ whose closure $\mathcal{A}^-$ in $\mathcal{J}^\delta$ is compact. Since we are dealing with meromorphic functions, uniform convergence on $\mathcal{A}$ means that in the spherical metric, dist $(f_n(z), f(z)) < \epsilon$ for $n > n_0(\epsilon)$ and $z \in \mathcal{A}$. By the definition of kernel, $f_n(z)$ is defined on $\mathcal{A}$ for $n > n_1$ and hence the definition of subuniform convergence on $\mathcal{J}^\delta$ makes sense. If $f_n(z) \to f(z)$ subuniformly on $\mathcal{J}^\delta$ then either $f(z)$ is meromorphic on $\mathcal{J}^\delta$ or $f(z) \equiv \infty$.

Now *we shall decree that* $f_n(z) \to f(z)$ *subuniformly on* $\mathcal{J}$ *if and only if* 1) $f_n(z) \to f(z)$ *subuniformly on* $\mathcal{J}^\delta$ *and* 2), *using the notation of* §25, *if* Q *is a branch point of* $\mathcal{J}$ *added to* $\mathcal{D}^\delta$ *then for any* $\epsilon > 0$ *there exist* $n_0(\epsilon)$ *and* $r_0(\epsilon)$ *such that, in the spherical metric,* dist $(f_n(z), f(Q)) < \epsilon$ *for* $n > n_0(\epsilon)$, $r < r_0(\epsilon)$ *and* $z \in \mathcal{B}_n$.

If $f_n(z) \to f(z)$ subuniformly on $\mathcal{J}$ then $f(z)$ is either $\equiv \infty$ or meromorphic on $\mathcal{J}$. For by 2) $f(z)$ is continuous, in the spherical metric, at any Q. Since $f(z)$ is either $\equiv \infty$ or meromorphic on $\mathcal{J}^\delta$ the conclusion follows.

Correspondingly if $f_n(z)$ is meromorphic on $[\mathcal{J}_n; P_n]$ and if $[\mathcal{J}; P]$ is a kernel of $\{\mathcal{J}_n; P_n\}$ we shall say that $\{f_n(z)\}$ is a **normal family** on $\mathcal{E} \subset \mathcal{J}$ if and only if every subsequence of $\{f_n\}$ contains a subsequence which converges subuniformly on $\mathcal{E}$.

29. We start now with the extension of Theorem 7.

THEOREM 10. *Let* $[\mathcal{J}_n; P_n]$ *be Riemann surfaces over* $\mathcal{Z}$, *each containing a schlicht disc of radius* r_0 *about* P_n *over* $z=a\neq\infty$. *Let* $[\mathcal{J}; P]$ *be a kernel of* $\{\mathcal{J}_n; P_n\}$. *Let* $[\mathcal{G}_n; Q_n]$ *be Riemann surfaces over* $\mathcal{W}$, *each containing a schlicht disc of radius* s_0 *about* Q_n *over* $w=b\neq\infty$. *Let* $f_n(z)$ *be meromorphic on* $\mathcal{J}_n$ *and let* $w=f_n(z)$ *map* $\mathcal{J}_n$ *one-one onto a subsurface of* $\mathcal{G}_n$ *with*

$$\text{dist}\,(f(P_n), Q_n) \to 0 \tag{20}$$

as $n\to\infty$, *the distance being measured on* $\mathcal{G}_n$. *Let* $f_n(z)\to f(z)$ *subuniformly on* $\mathcal{J}$ *with*

$$f'(a) = c \neq 0, \infty, \qquad \textit{at } P \textit{ over } z = a. \tag{21}$$

Then there exists a kernel $[\mathcal{G}; Q]$ *of* $\{\mathcal{G}_n; Q_n\}$ *and a subsurface* $[\mathcal{K}; Q]\subset[\mathcal{G}; Q]$ *such that* $w=f(z)$ *maps* $\mathcal{J}$ *one-one onto* $\mathcal{K}$ *with* $f(P)=Q$.

In general the kernel $\mathcal{G}$ will vary for different choices of kernels $\mathcal{J}$. Consider for example any case where there is more than one kernel $\mathcal{J}$ and $f_n(z)=z$.

We first prove Theorem 10 under the additional hypotheses: $\mathcal{J}_n$ has no branch points and $f_n(z)$ is locally schlicht on $\mathcal{J}_n$, i.e., if A is any point of $\mathcal{J}_n$ then there exists a neighborhood of A in which $f_n(z)$ is schlicht. If A lies over a finite point of $\mathcal{Z}$ and is not a pole of $f_n(z)$ then this condition is equivalent to $f_n'(z)\neq 0$ at A. The corresponding conditions at other points A are apparent.

It follows from (20) and (21) that $f(z)$ is meromorphic on $\mathcal{J}$ and non-constant. As a limit of locally schlicht functions, $f(z)$ is locally schlicht on $\mathcal{J}$ and $w=f(z)$ maps $\mathcal{J}$ onto an unbranched Riemann surface $\mathcal{K}$ over $\mathcal{W}$, with $f(P)=Q$. By §24 we need only show that $[\mathcal{K}, Q]$ satisfies 1° of §22. Let $\mathcal{B}$ be any subsurface of $\mathcal{H}$, containing Q, whose closure $\mathcal{B}^-$ in $\mathcal{H}$ is compact. Let $\mathcal{B}$ be the image of $\mathcal{A}\subset\mathcal{J}$ under $w=f(z)$. Then $\mathcal{A}^-$ in $\mathcal{J}$ is compact. We must show that $w=f(z)$ maps $\mathcal{A}$ into $\mathcal{G}_n$ for n sufficiently large.

Since $f(z)$ is locally schlicht, each point of $\mathcal{J}$ is contained in a schlicht neighborhood U on $\mathcal{J}$: $\{|z-\alpha|<r\}$ (or $\{|z|>M\}$) such that the schlicht disc

$$U^* = \{|z-\alpha|<2r\}, \qquad (\text{or } \{|z|>M/2\}), \tag{22}$$

containing U, is in $\mathcal{J}$ and such that $f(z)$ and $f_n(z)$ are schlicht in U^* for $n>n_0(U)$.

Now $\mathcal{A}^-$ is covered by a finite number of such neighborhoods: $\mathcal{A}^-\subset\bigcup_1^N U_k$. We may assume that $P\in U_1$ and that $\bigcup_1^N U_k$ is connected. We shall prove that $w=f(z)$ maps $\bigcup_1^N U_k$, and hence $\mathcal{A}^-$ into $\mathcal{G}_n$ for $n>n_0$. Let $w=f_n(z)$ map U_i and U_i^* onto $V_{i,n}$ and $V_{i,n}^*$. Then by Theorem 6 $V_{i,n}$ and $V_{i,n}^*$ converge (as plane domains) to their kernels V_i and V_i^*, and $w=f(z)$ maps U_i and U_i^* onto V_i and V_i^*. Now V_i^- is the image of a compact set in U_i^* and hence $V_i\subset V_i^-\subset V_{i,n}^*$ for $n>n_i$. Now, not as plane domains, but on the Riemann surfaces, $w=f_n(z)$ maps $\bigcup_1^N U_k^*$ onto $\bigcup_1^N V_{k,n}^*\subset\mathcal{G}_n$, and $w=f(z)$ maps $\bigcup_1^N U_k$ onto $\bigcup_1^N V_k\subset\bigcup_1^N V_{k,n}^*$ for $n>\max(n_1, n_2, \cdots, n_N)$. This completes the proof of the special case of Theorem 10.

30. In the general case let $\mathcal{J}^\delta$ be the deleted kernel (cf. §25) corresponding to $\mathcal{J}$. $\mathcal{J}^\delta$ is a kernel of $\{\mathcal{J}_n^\delta\}$. Let $\mathcal{J}_n^{\delta*}$, $(\mathcal{J}^{\delta*})$ denote $\mathcal{J}_n^\delta$, $(\mathcal{J}^\delta)$ with all the (isolated) points at which $f_n(z)$, $(f(z))$ is not locally schlicht deleted. By Hurwitz' theorem [28, p. 6], $\mathcal{J}^{\delta*}$ is a kernel of $\{\mathcal{J}_n^{\delta*}\}$ and by the special case of Theorem 10 which we have already proved, $w=f(z)$ maps $\mathcal{J}^{\delta*}$ into a kernel $\mathcal{G}$.

Let P' over z_0 be a point of $\mathcal{J}^\delta$ at which $f(z)$ is not locally schlicht. Then there is a schlicht neighborhood U' of P', over $|z-z_0|<r$, contained in $\mathcal{J}_n^\delta$ for $n>n_0$, and $w=f(z)$ maps the corresponding deleted neighborhood U^* on $\mathcal{J}^\delta$ onto a deleted branch neighborhood on $\mathcal{G}^\delta$. But the associated branch point is actually a point of $\mathcal{G}$, since $U'\subset\mathcal{J}_n^\delta$ and $w=f_n(z)$ maps $\mathcal{J}_n^\delta$ into $\mathcal{G}_n$ and therefore maps U' onto the required simply connected $\mathcal{B}_n$ (cf. §25). Hence $w=f(z)$ maps $\mathcal{J}^{\delta*}\cup\{P'\}$ into G and thus maps $\mathcal{J}^\delta$ into $\mathcal{G}$.

A similar argument disposes of the branch points of $\mathcal{J}$. Let P'' be a branch point of $\mathcal{J}$ over $z=z_1$. Let U be the neighborhood over $|z-z_1|<r$, and let U^* be U with P'' deleted. If U_n is the corresponding neighborhood in $\mathcal{J}_n$ over $|z-z_1|<2r$ in which any compact part of U^* is embedded, then for r small enough U_n is simply connected (cf. §25). Let $f(U^*)=V^*\subset\mathcal{G}$ and $f_n(U_n)=V_n\subset\mathcal{G}_n$. Then V_n is simply connected and covers V^*— for $n>n_0$. Since (cf. §28) $f(z)$ is continuous at P'', $f(P'')\in V_n \subset\mathcal{G}_n$. Since $V^*\subset\mathcal{G}$ it follows that $V\subset\mathcal{G}$ or $f(P'')\in\mathcal{G}$. Thus $w=f(z)$ maps $\mathcal{J}$ one-one onto a subsurface of a kernel $\mathcal{G}$, which completes the proof of Theorem 10.

31. The extension of Theorem 6 is the following.

THEOREM 11. *Let* $[\mathcal{J}_n;\, P_n]$ *be Riemann surfaces over* $\mathcal{Z}$, *each containing a schlicht disc of radius* r_0 *about* P_n *over* $z=a\neq\infty$. *Let* $[\mathcal{J}_n;\, P_n]\rightarrow[\mathcal{J};\, P]$.

Let $[\mathcal{G}_n;\, Q_n]$ *be Riemann surfaces over* $\mathcal{W}$, *each containing a schlicht disc of radius* s_0 *about* Q_n *over* $w=b\neq\infty$.

Let $f_n(z)$ *be meromorphic on* $\mathcal{J}_n$ *and let* $w=f_n(z)$ *map* $\mathcal{J}_n$ *one-one onto* $\mathcal{G}_n$ *with*

$$\text{dist}\,(f_n(P_n),\, Q_n)\rightarrow 0 \qquad (n\rightarrow\infty), \tag{23}$$

and

$$\lim_{n\rightarrow\infty} f_n'(a) = c\neq 0,\ \infty \qquad \textit{at } P_n \textit{ over } a. \tag{24}$$

Let $\phi_n(w)$ *be the function inverse to* $f_n(z)$, *and let* $\{\phi_n(w)\}$ *form a normal family on any kernel* $[\mathcal{G};\, Q]$ *of* $\{\mathcal{G}_n;\, Q_n\}$.

Then the following two conclusions are valid:

A). *If* $f_n(z)\rightarrow f(z)$ *subuniformly on* $\mathcal{J}$, *then* $[\mathcal{G}_n;\, Q_n]\rightarrow[\mathcal{G};\, Q]$ *and* $w=f(z)$ *maps* $\mathcal{J}$ *one-one onto* $\mathcal{G}$ *with* $Q=f(P)$ *and* $f'(a)=c$ *at* P. *Also* $\phi_n(w)\rightarrow\phi(w)$ *subuniformly on* $\mathcal{G}$ *and* ϕ *is the inverse of* f.

B). *If* $\{f_n\}$ *is a normal family on* $\mathcal{J}$, *if* $[\mathcal{G}_n;\, Q_n]\rightarrow[\mathcal{G};\, Q]$, *and if* $\mathcal{J}$ *is, topologically, not a sphere, then* $f_n(z)\rightarrow f(z)$ *subuniformly on* $\mathcal{J}$ *and* $w=f(z)$ *maps* $\mathcal{J}$ *one-one onto* $\mathcal{G}$. *Also* $\phi_n(w)\rightarrow\phi(w)$ *subuniformly on* $\mathcal{G}$ *and* ϕ *is the inverse of* f.

32. Corollaries. *The condition that* $\{\phi_n(w)\}$ *be normal on any kernel* $\mathcal{G}$ *may be dropped provided either one of the following con-*

ditions holds:

α). There exists a fixed Riemann surface $[\mathcal{A}; T]$ *over* $\mathcal{Z}$ *such that* $[\mathcal{J}_n; P_n] \subset [\mathcal{A}; T]$ *for* $n > n_0$. *In particular this condition is fulfilled if the* $\mathcal{J}_n$ *are domains on* $\mathcal{Z}$ *or if the sequence* $\{\mathcal{J}_n; P_n\}$ *is monotonic.*

β). There exists a constant $N < \infty$ *and three distinct points* z_1, z_2, z_3 *of* $\mathcal{Z}$ *such that, for all* n, *there is no point of* $\mathcal{J}_n$ *over* z_1 *and at most* N *distinct points of* $\mathcal{J}_n$ *over* z_2 *and* z_3. *The significance of "distinct" here is that a branch point over* z_2 *or* z_3 *is counted as a single point.*

We shall prove Theorem 11 and then prove the corollaries.

33. Proof of A. It follows from Theorem 10 that there exists a kernel $\mathcal{G}$ of $\{\mathcal{G}_n\}$ such that $w=f(z)$ maps $\mathcal{J}$ into $\mathcal{G}$. If $\mathcal{H}$ is any kernel of any subsequence $\{\mathcal{G}_{\mu_n}\}$ then by the normality of $\{\phi_n\}$ there is a subsequence, $\{\lambda_n\}$, of $\{\mu_n\}$ such that $\phi_{\lambda_n} \to \phi$ subuniformly on $\mathcal{H}$ and, by Theorem 10, $z=\phi(w)$ maps $\mathcal{H}$ into $\mathcal{J}$. The composite function $f(\phi(w))$ maps $\mathcal{H}$ into $\mathcal{G}$. But on any domain $\mathcal{A} \subset \mathcal{H}$ whose closure in $\mathcal{H}$ is compact, $f_{\lambda_n}(\phi_{\lambda_n}(w))$ is defined and $\equiv w$. Hence $f(\phi(w)) \equiv w$ and therefore $\mathcal{H} = \mathcal{G}$. For a proper inclusion $\mathcal{H}^\delta \subset \mathcal{G}^\delta$ would be nonsense by the maximal property of deleted kernels, and hence $\mathcal{H}^\delta = \mathcal{G}^\delta$. From the definition of kernel, $\mathcal{H} \supset \mathcal{G}$, and since $\mathcal{H} \subset \mathcal{G}$, we must have $\mathcal{H} = \mathcal{G}$. It then follows from the specification of $\mathcal{H}$ that $\mathcal{G}_n \to \mathcal{G}$.

Also $w=f(z)$ maps $\mathcal{J}$ *onto* $\mathcal{G}$ for a proper inclusion $f(\mathcal{J}) \subset \mathcal{G}$ would imply that the identical map of $\mathcal{H}$ *onto* $\mathcal{G}$ by $w=f(\phi(w))$ were strictly into which is impossible.

If $\mu_n = n$ then ϕ may be any limit function of the normal family $\{\phi_n\}$ on $\mathcal{G}$. As we have seen, the only such limit function is the inverse of f and hence $\phi_n \to \phi$ subuniformly on $\mathcal{G}$. This completes the proof of A.

34. Proof of B. Let $\{f_{\lambda_n}\}$ be a subuniformly convergent subsequence of $\{f_n\}$ with $f_{\lambda_n} \to f$. By A, $w=f(z)$ maps $\mathcal{J}$, which is *the* kernel of $\{\mathcal{J}_{\lambda_n}\}$, onto $\mathcal{G}$, which is *the* kernel of $\{\mathcal{G}_{\lambda_n}\}$. If g is any limit function of the normal family $\{f_n\}$ on $\mathcal{J}$, then g has the same mapping property as f and $g^{-1}(f(z))$ maps $\mathcal{J}$ onto itself with P fixed and derivative 1 at P. By I Theorem 4 this map is the identity map and hence $g=f$. Therefore $f_n \to f$ subuniformly on $\mathcal{J}$.

The rest of B, namely that $\phi_n \rightarrow \phi$ etc. now follows from A. This completes the proof of Theorem 11.

35. Proof of Corollary α. The proof amounts to showing that the normality of $\{\phi_n\}$ on $\mathcal{G}$ follows from the existence of $\mathcal{A}$ and (23) and (24). Slightly more generally we have the following theorem.

THEOREM 12. *Let* $[\mathcal{G}_n; Q_n]$ *be Riemann surfaces over* $\mathcal{W}$, *each containing a schlicht disc of radius* s_0 *about* Q_n *over* $w=b$. *Let* $[\mathcal{G}; Q]$ *be a kernel of* $\{\mathcal{G}_n; Q_n\}$. *Let* $z=\phi_n(w)$ *be meromorphic on* $\mathcal{G}_n$, *holomorphic in* $\{|w-b|<s_0\}$ *about* Q_n, *and map* $\mathcal{G}_n$ *one-one onto a subsurface of a fixed Riemann surface* $\mathcal{A}$ *over* $\mathcal{Z}$. *Let* $\mathcal{K}$ *be a compact subset of* $\mathcal{A}$, *containing no point over* $z=\infty$, *such that* $\phi_n(Q_n)\in\mathcal{K}$. *Let*

$$\frac{1}{k} \leqq |\phi_n'(b)| \leqq k, \qquad \text{at } Q_n, \tag{25}$$

where k *is a positive constant.*

Then $\{\phi_n\}$ *is normal on* $\mathcal{G}$ *and any limit function* $\phi(w)$ *is such that* $z=\phi(w)$ *maps* $\mathcal{G}$ *one-one onto a subsurface of* $\mathcal{A}$.

Remark. In applying Theorem 12 to prove Corollary α, the hypotheses of Theorem 12 are obviously fulfilled except for (25) and the fact that $\phi_n(w)$ is holomorphic in a disc $\{|w-b|<s_0\}$ about Q_n. We show that these conditions are also satisfied. Now $\{f_n\}$ is normal on $\mathcal{J}$ in both part A and part B of Theorem 11. By (23), $\{f_n\}$ is uniformly bounded at P, hence is uniformly bounded in the neighborhood U of P over $|z-a|<r<r_0$ for r small enough, and therefore is holomorphic in U. Since $\{f_n(z)\}$ is uniformly bounded it follows from (24) that $f_n(z)$ is schlicht in U for r small enough. Then by I Theorem 1 the image of U will cover a disc about $f_n(P_n)$ of radius $|c|r/8$ for $n>n_0$. By (23), Q_n will lie within the disc about $f_n(P_n)$ with radius $|c|r/16$ for $n>n_1$. Hence for such n, $\phi_n(w)$ will be holomorphic and schlicht in the disc of radius $s_0=|c|r/16$ about Q_n. It also follows from I Theorem 1 that if $f_n(P_n')=Q_n$, then $P_n'\rightarrow P$ and $f_n'(P_n')\rightarrow c$, or $\phi_n'(Q_n)\rightarrow 1/c$, whence (25).

Proof. We prove first that $\{\phi_n\}$ is normal on $\mathcal{G}^\delta$. For this purpose it is sufficient to show that $\{\phi_n\}$ is normal on any simply connected subdomain $\mathcal{H} \subset \mathcal{G}^\delta$, whose closure $\mathcal{H}^-$ in $\mathcal{G}^\delta$ is compact, and such that $Q \in \mathcal{H}$. The normality on $\mathcal{G}^\delta$ then follows in the standard fashion by means of the diagonal process.

Since we must prove that any subsequence of $\{\phi_n\}$ has a subsequence which converges subuniformly on $\mathcal{H}$ we may assume to start with that $\phi_n(Q_n) \to P \in \mathcal{K}$, P over $z=a$, and that $\phi_n'(Q_n) \to \gamma \neq 0, \infty$.

Let $\zeta = g(z)$ map the universal covering of $\mathcal{A}$ onto either $|\zeta| < R \leqq \infty$ or the complete ζ-sphere so that for a particular determination, $g_1(z)$, of $g(z)$ at P, $g_1(a)=0$ and $g_1'(a)=1$. In the elliptic case $\mathcal{A}$ is an algebraic surface of genus zero and we may assume that the pole of $g(z)$ lies at one of the points of $\mathcal{A}$ over $z=\infty$.

Let $\zeta = g(\phi_n(w)) = \psi_n(w)$ be defined by $\psi_n(w) = g_1(\phi_n(w))$ at Q and analytic continuation. Since $\mathcal{H}^- \subset \mathcal{G}_n$ for $n > n_0$ and since $\mathcal{H}$ is simply connected, $\psi_n(w)$ is defined and uniform on $\mathcal{H}$ for $n > n_0$. Also $\psi_n(w)$ is schlicht on $\mathcal{H}$ and $\psi_n(Q) \to 0$ and $\psi_n'(Q) \to \gamma$. Since $\phi_n(w)$ is holomorphic in $\{|w-b| < s_0\}$ about Q, $\psi_n(w)$ is holomorphic in $\{|w-b| < s_0\}$ about Q; it is here that we use the specified normalization of g in the elliptic case. It follows now that $\{\psi_n(w)\}$ is normal on $\mathcal{H}$, for by I Theorem 1, $\psi_n(w)$ is subuniformly bounded in $\{|w-b| < s_0\}$ about Q, and maps $\{|w-b| < s_0/2\}$ onto a domain covering a fixed neighborhood of $\zeta = 0$. Hence outside of $\{|w-b| < s_0/2\}$ about Q, $\psi_n(w)$ is uniformly bounded away from 0. Hence $\{\psi_n\}$ is normal on $\mathcal{H}$. Let $\psi_{\lambda_n} \to \psi$ subuniformly on $\mathcal{H}$. Since $\psi_n'(Q) \to \gamma \neq 0, \infty$, $\zeta = \psi(w)$ is meromorphic, non-constant, schlicht on $\mathcal{H}$, and maps $\mathcal{H}$ onto a subset of $|\zeta| < R$ on which the inverse function, g^{-1}, of g is defined. It follows that $g^{-1}(\psi_n(w)) = \phi_n(w)$ converges subuniformly on $\mathcal{H}$ to $g^{-1}(\psi(w)) = \phi(w)$.

Now any kernel of the surfaces $\phi_n(G_n)$ is contained in $\mathcal{A}$ and therefore $z = \phi(w)$ maps $\mathcal{G}^\delta$ one-one onto a subsurface of $\mathcal{A}$ by virtue of Theorem 10.

36. To reinstate the branch points and prove normality on $\mathcal{G}$ it will suffice to show that if $\phi_n \to \phi$ subuniformly on $\mathcal{G}^\delta$, then ϕ may be defined on $\mathcal{G}$ so that $\phi_n \to \phi$ subuniformly on $\mathcal{G}$. Let P' be any branch point of $\mathcal{G}$ over $w = w_0$, let V_r be that component of $\mathcal{G}$ over $\{|w-w_0|<r\}$ containing P' and let V_r^δ be V_r with P' removed. Let $V_{r,n}$ be the component of $\mathcal{G}_n$ over $\{|w-w_0|<r\}$ corresponding to V_r, i.e., in defining $\mathcal{G}$, any compact subset of V_r^δ is embedded in $V_{r,n}$. By §25, $V_{r,n}$ is simply connected for r small enough and $z=\phi_n(w)$ maps $V_{r,n}$ onto a simply connected domain $U_{r,n}\subset\mathcal{A}$. Let $z=\phi(w)$ map V_r^δ onto $U_r^\delta\subset\mathcal{A}$. Now $z=\phi(w)$ maps any compact set on V_r^δ into $U_{2r,n}$ for $n>n_0(r)$ and hence, by the simple connectivity of $U_{2r,n}$,

$$U_r^\delta\subset U_{2r,n}, \qquad n>n_0(r).$$

Now $\phi_n(w)$ is meromorphic on $V_{2r,n}^-$ for r small enough and therefore $U_{2r,n}$ is finite sheeted over Z. Hence $\phi(w)$ has a removable rather than an essential singularity at P' and U_r^δ must be a deleted neighborhood on $\mathcal{A}$. By the simple connectivity of $U_{2r,n}$, $z=\phi(w)$ maps V_r onto $U_r\subset U_{2r,n}$. Thus $z=\phi(w)$ is defined on $\mathcal{G}$ and maps $\mathcal{G}$ onto a subsurface of $\mathcal{A}$.

The normality of $\{\phi_n\}$ on $\mathcal{G}$ involves more than the fact that $\phi(w)$ may be defined on all $\mathcal{G}$, but this normality now follows easily for diam $U_{2r}\to 0$ as $r\to 0$ and $U_{r,n}\subset U_{2r}$ for $n>n_1(r)$. Then by §28 we see that $\{\phi_n\}$ converges subuniformly in a neighborhood of P'. Hence $\{\phi_n\}$ is normal on $\mathcal{G}$ and we have completed the proof of Theorem 12.

37. Proof of Corollary β. As with α) we shall show that the hypothesis β) entails the normality of $\{\phi_n\}$ on $\mathcal{G}$, provided the remaining hypotheses of Theorem 11 are satisfied. Now $\{\phi_n\}$ is quasi-normal [14] on $\mathcal{G}^\delta$; that is, any subsequence of $\{\phi_n\}$ contains a subsequence which converges subuniformly on $\mathcal{G}^*$ where $\mathcal{G}^*$ is obtained by removing a finite number of points from $\mathcal{G}^\delta$. This fact may be seen as follows. Any subsequence of $\{\phi_n\}$ contains a subsequence $\{\phi_{\lambda_n}\}$ such that the points of $\mathcal{G}^\delta$ where $\phi_{\lambda_n}(w)=z_2$ or z_3 tend to points $T_1, T_2, \cdots, T_k$ $(0\leqq k\leqq N)$ of $\mathcal{G}^\delta$ and (possibly) to the "ideal boundary" of $\mathcal{G}^\delta$. Then the $\mathcal{G}^*$ associated with $\{\phi_{\lambda_n}\}$ is obtained by removing the k points T_j from $\mathcal{G}^\delta$. Clearly $\phi_{\lambda_n}(w)$ does not assume the values z_1, z_2, z_3 on any compact subset $\mathcal{A}$ of $\mathcal{G}^*$ for $n>n(\mathcal{A})$ and hence $\{\phi_{\lambda_n}\}$ is normal on $\mathcal{G}^*$. Let $\{\phi_{\mu_n}\}$ be a subsequence of $\{\phi_{\lambda_n}\}$ such that

$$\phi_{\mu_n}(w)\to\phi(w)$$

subuniformly on $\mathcal{G}^*$.

By the remark following Theorem 12, $\phi_n(w)$ is schlicht and holomorphic in a disc V about $Q \in \mathcal{G}$. Together with (23) and (24) this fact implies that $\{\phi_n\}$ is normal in V and that no limit function of $\{\phi_n\}$ is constant in V. We may assume then that $V \subset \mathcal{G}^*$, for any T_j which is a point of V need not be removed. Also $\phi(w)$ is not a constant and hence $\phi(w) \neq z_1$ on $\mathcal{G}^*$.

Let S_j be a schlicht neighborhood of T_j, bounded by a circle γ_j, such that $S_j^- = S_j \cup \gamma_j \subset \mathcal{G}^\delta$. Then if $z_1 \neq \infty$, $\psi_{\mu_n}(w) = 1/(\phi_{\mu_n}(w) - z_1)$ is holomorphic in S_j^- and converges to $\psi(w) = 1/(\phi(w) - z_1)$ on γ_j; if $z_1 = \infty$ we use $\psi = \phi$. Since $\psi(w)$ is bounded on γ_j it follows from the maximum principle that $\psi_{\mu_n}(w)$ is uniformly bounded in S_j^- for $n > n_0$. Therefore $\{\psi_{\mu_n}\}$ is normal in S_j. Then $\{\phi_{\mu_n}\}$ is also normal in S_j and since $\phi_{\mu_n} \to \phi$ at all points of S_j except T_j, the definition of ϕ may be extended so that $\phi_{\mu_n} \to \phi$ uniformly in S_j. Therefore $\phi_{\mu_n} \to \phi$ subuniformly on $\mathcal{G}^\delta$.

Now if P' is any branch point of $\mathcal{G}$ over $w = w_0$, let V_r, V_r^δ, $V_{r,n}$, $U_r = \phi(V_r)$ and $U_{r,n} = \phi(V_{r,n})$ be as in §36. To simplify the labelling we shall assume that $\mu_n = n$ and $\phi_n \to \phi$ subuniformly on $\mathcal{G}^\delta$. Since $\phi \neq z_i$ $(i = 1, 2, 3)$ except at a finite number of points on $\mathcal{G}^\delta$, $\phi \neq z_i$ $(i = 1, 2, 3)$ on V_r^δ for r sufficiently small. Therefore ϕ has a removable singularity at P' and we extend ϕ to all of $\mathcal{G}$ accordingly. Also

$$\text{(26)} \qquad \operatorname{diam}(U_r) \to 0 \qquad \text{as } r \to 0,$$

where diam (U_r) denotes the diameter, measured in the spherical metric, of the projection of U_r into $\mathcal{Z}$.

Let γ_r and $\gamma_{r,n}$ be the boundaries of V_r, and $V_{r,n}$ respectively. If $\phi(w) \neq z_4$ on γ_r, then

$$\text{(27)} \qquad \underset{\gamma_r}{\Delta \arg} \frac{\phi - z_1}{\phi - z_4} = \underset{\gamma_{r,n}}{\Delta \arg} \frac{\phi_n - z_1}{\phi_n - z_4},$$

for $n > n_1$, which is a consequence of the fact that $\phi_n \to \phi$ on γ_r. Also it is clear that (27) holds for $n > n_1(r, \epsilon)$ and all z_4 such that dist $(z_4, U_r) > \epsilon > 0$. For any such z_4, (27) becomes: (number of z_1-points of ϕ in V_r) = −(number of z_4-points of ϕ_n in $V_{r,n}$)

since $\phi_n \neq z_1$. But the two parentheses in this equation are non-negative and hence each is zero. From the first we obtain: $\phi \neq z_1$ on $\mathcal{G}$, and from the second:

$$\operatorname{diam}(U_{r,n}) \leqq \operatorname{diam}(U_r) + 2\epsilon, \qquad n > n_1(r, \epsilon).$$

By (26), diam $(U_r) < \epsilon$ for $r \leqq r_0$, and hence diam $(U_{r,n}) \leqq$ diam $(U_{r_0}, n) \leqq$ diam $(U_{r_0}) + 2\epsilon < 3\epsilon$, or

$$\operatorname{diam}(U_{r,n}) < 3\epsilon, \qquad n > n_2(\epsilon),\ r < r_0(\epsilon).$$

By §28 it follows that $\phi_n \to \phi$ uniformly in a neighborhood of P' and hence $\phi_n \to \phi$ subuniformly on $\mathcal{G}$. This result completes the proof of Corollary β.

It should be noted that we have used the fact that the branch point P' is actually a point of $\mathcal{G}$ in deriving (27); namely, $V_{r,n}$ is simply connected and the complete boundary of $V_{r,n}$ is $\gamma_{r,n}$, which corresponds to γ_r.

38. Remarks. It is of course possible to generalize Corollary β to the case where $z_i = z_{i,n}$ varies with n, provided there exists a constant $k > 0$ such that, in the spherical metric,

$$\operatorname{dist}(z_{i,n}, z_{j,n}) \geqq k, \qquad i \neq j,\ n \geqq 1.$$

The proof of this generalization is easily reduced to Corollary β by means of the transformations

$$\psi_n(w) = \frac{(z_{3,n} - z_{1,n})(\phi_n(w) - z_{2,n})}{(z_{3,n} - z_{2,n})(\phi_n(w) - z_{1,n})}.$$

The following simple example throws some light on Theorem 11. Let $\mathcal{J}_n$ over Z be two sheeted with two first order branch points over $z = n$ and $z = \infty$; let P_n be one of the points of $\mathcal{J}_n$ over $z = 0$. Let $\mathcal{G}_n = \mathcal{W}$ and let Q_n be the point $w = 0$. Clearly $[\mathcal{J}_n; P_n] \to \mathcal{J} =$ the finite z-plane, and $\mathcal{G}_n \to \mathcal{G} = \mathcal{W}$. Let

$$w = f_n(z) = 2n - 2\sqrt{n(n - z)},$$

$$z = \phi_n(w) = w - w^2/4n.$$

Clearly all the hypotheses of Theorem 11, part A and part B are satisfied, except for the normality of $\{\phi_n\}$ on $\mathcal{W}$. In particular: $f_n'(0)=1$ at P_n,

$$f_n(z) \to f(z) = z, \qquad \text{subuniformly on } \mathcal{J},$$

and $\phi_n(w) \to w$ subuniformly on the finite w-plane but *not* on $\mathcal{G}$. Now $w=f(z)$ maps $\mathcal{J}$ into $\mathcal{G}$, not *onto*, and hence the hypothesis that $\{\phi_n\}$ be normal on $\mathcal{G}$ is essential to both part A and part B of Theorem 11.

This same example shows that in Corollary β it would not do to allow a finite number of points over z_1.

If we switch the roles of z, w and $\mathcal{J}_n$, $\mathcal{G}_n$ and make $\mathcal{G}$ not a sphere by removing a fixed point, say $w=1$, from all $\mathcal{G}_n$ and removing the corresponding points from the $\mathcal{J}_n$, then we see that the requirement that $\{f_n\}$ be normal on $\mathcal{J}$ is essential in part B of Theorem 11.

III. REAL ENTIRE FUNCTIONS AND RIEMANN SURFACES

1. The class W_n. *The Riemann surface $\mathcal{G}$ over $\mathcal{W}$ belongs to W_n if and only if $\mathcal{G}$ is the image of $\mathcal{Z}$ by a polynomial $w=P(z)$ of degree $n+1$ such that*

$$P'(z) = \prod_{\nu=1}^{n}\left(1 - \frac{z}{b_\nu}\right), \tag{1}$$

$$0 < b_1 \leqq b_2 \leqq \cdots \leqq b_n, \tag{2}$$

and

$$P(0) = 0. \tag{3}$$

Also let

$$a_\nu = P(b_\nu), \qquad 1 \leqq \nu \leqq n. \tag{4}$$

It is a simple matter to construct any such $\mathcal{G}$ by the well-known process of considering the points of $\mathcal{Z}$ where $P(z)$ is real. These points constitute the real axis and n Jordan curves C_ν, on $\mathcal{Z}$, symmetric about the real axis, C_ν intersecting the real axis at b_ν, and such that all C_ν are disjoint except for $z=\infty$ which is common to all C_ν, and some of the points b_ν if not all the inequalities in (2) are strict. Let the $n+1$ disjoint domains of $\mathcal{Z}$ determined by the C_ν be D_ν, $1 \leqq \nu \leqq n+1$, where D_ν is the domain between $C_{\nu-1}$ and C_ν (and containing the interval $(b_{\nu-1}, b_\nu)$ of the real axis) for $1<\nu<n+1$, D_1 is bounded by C_1 alone and D_{n+1} is bounded by C_n alone. Let $D_\nu^+ (D_\nu^-)$ denote that part of D_ν which lies in the half-plane $\Im z>0$ ($\Im z<0$). By applying the theorem of Darboux [19, pp. 397–401] to each $D_\nu^\pm$ the structure of $\mathcal{G}$ becomes obvious. Namely, $\mathcal{G}$ consists of $n+1$ sheets S_ν, $1 \leqq \nu \leqq n+1$, where $w=P(z)$ maps D_ν schlichtly onto S_ν. For $1<\nu<n+1$, S_ν is $\mathcal{W}$ cut along the real axis from $a_{\nu-1}$ to $(-)^\nu\infty$ and from a_ν to $(-)^{\nu+1}\infty$. For $\nu=1$ and $n+1$ there is a single cut: from a_1 to $+\infty$ and from a_n to $(-)^{n+1}\infty$ respectively. The connection of the various sheets is determined by the manner in which the D_ν border one another on $\mathcal{Z}$. Namely, S_ν and $S_{\nu+1}$ are joined "criss-cross" along the corresponding cuts issuing

from a_ν. As a result, $\mathcal{G}$ is $(n+1)$-sheeted with a total branch order of $2n$. In general, when the inequalities in (2) are all strict, $\mathcal{G}$ possesses n first-order branch points and one branch point of order n (over ∞). Also, since $\mathcal{G}$ is the image of $\mathcal{Z}$, $\mathcal{G}$ is closed and of genus zero. The above construction of $\mathcal{G}$ implies that

$$(5) \qquad 0 < a_1, \qquad a_{2\nu} \leqq a_{2\nu \pm 1}, \qquad 1 \leqq \nu \leqq \frac{n}{2}.$$

In other words, the sequence $\{a_\nu\}$ oscillates regularly. This fact is also directly apparent from (1) and (2).

Conversely, any $\mathcal{G}$ *which consists of* $n+1$ *sheets* S_ν *hung together along the cuts specified above, where* a_ν *satisfy* (5), *is of class* W_n. Because of (5) this construction of $\mathcal{G}$ is possible. The resulting $\mathcal{G}$ is clearly closed and of genus zero, with a single point over infinity. Hence $\mathcal{G}$ is the image of $\mathcal{Z}$ by a polynomial $P(z)$. By normalizing this map properly, $P(z)$ may be made to satisfy (1), (2), and (3). Namely, by the symmetry of $\mathcal{G}$, the b_ν will lie along a ray (which may be made the positive real axis) if $\infty \leftrightarrow \infty$ and $P(0) = 0 \in S_1$. Finally we make $P'(0) = 1$ by the proper choice of a unit in the z-plane.

Because of this normalization: (2), (3), and $P'(0) = 1$, which is a consequence of (1), it is clear that the correspondence between W_n and all polynomials satisfying (1), (2), and (3), is one-one.

2. The Classes W_n^k, $W_n^{k,\pm}$. *The Riemann surface* $\mathcal{G}$ *over* $\mathcal{W}$ *belongs to* W_n^k, $(n \geqq 0, k \geqq 0)$ *if and only if* $\mathcal{G}$ *is the image of the z-plane by an entire function* $w = f(z)$ *of the form*

$$(6) \qquad f'(z) = e^{\pi(z)} \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_\nu}\right),$$

$$(7) \qquad f(0) = 0,$$

$$(8) \qquad 0 < b_1 \leqq b_2 \leqq \cdots \leqq b_n,$$

and

$$(9) \qquad \pi(z) = c_1 z + c_2 z^2 + \cdots + c_k z^k, \qquad c_j \text{ real}, \ c_k \neq 0.$$

W_n^k *is split into two subclasses as follows:* $W_n^{k,+}$ $(W_n^{k,-})$ *consists* *of those surfaces in* W_n^k *for which* $c_k > 0$ ($c_k < 0$). The reason for distinguishing between these two subclasses will appear in §19. If $k = 0$ we interpret (9) as $\pi(z) \equiv 0$; then $W_n^0 = W_n$.

In order to make the correspondence between W_n^k and the functions of the form (6, 7, 8, 9) one-one we specify the point $P_0 \in \mathcal{G}$, over $w = 0$, corresponding to $z = 0$. As may readily be shown by example, when n is odd it may sometimes happen that two distinct functions of the form (6, 7, 8, 9) will give the same surface $\mathcal{G}$ with two different points corresponding to $z = 0$. Thus we decree that the elements of W_n^k shall be $[\mathcal{G}; P_0]$, not simply $\mathcal{G}$.

It should be noted that these classes of Riemann surfaces, as well as various classes to be defined later, constitute a classification of certain simple Riemann surfaces $\mathcal{G}$ as *definite* coverings of $\mathcal{W}$, but are meaningless for $\mathcal{G}$ considered intrinsically.

3. We next obtain the characteristic properties of the surfaces in W_n^k as coverings of $\mathcal{W}$. Let $\mathcal{G}$ be such a surface, associated with (6, 7, 8, 9) and set

$$a_\nu = f(b_\nu), \qquad 1 \leqq \nu \leqq n. \tag{10}$$

The sole algebraic branch points of $\mathcal{G}$ lie over $w = a_\nu$, $1 \leqq \nu \leqq n$, and if the inequalities in (8) are all strict there is a single first-order branch point over each a_ν. Also it is immediate by (6) and (8) that

$$0 < a_1, \qquad a_{2\nu} \leqq a_{2\nu \pm 1}, \qquad 1 \leqq \nu \leqq n/2. \tag{11}$$

Now

$$\left(1 + \frac{\pi(z)}{N}\right)^N \to e^{\pi(z)} \tag{12}$$

subuniformly in the z-plane. Let $f_N(z)$ be defined by

$$f_N'(z) = \left(1 + \frac{\pi(z)}{N}\right)^N \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_\nu}\right); \qquad f_N(0) = 0. \tag{13}$$

Set $\mathcal{J}_N = \mathcal{J} = z$-plane, with $P_N = 0$. Let $\mathcal{G}_N$ be the image of $\mathcal{J}_N$ by $w = f_N(z)$ with $f_N(P_N) = Q_N$ over $w = 0$. By (12),

$$f_N(z) \to f(z) \tag{14}$$

subuniformly on $\mathcal{J}$. Since $f'(0)=1$, there exists $r_0>0$ such that $f(z)$ is schlicht in $|z|<r_0$. By (14) and I Theorem 1, $\mathcal{G}_N$ will contain a schlicht disc of radius $r_0/8$ and center Q_N, for $N>N_0$. Hence all the conditions of II Theorem 11, Corollary α are fulfilled and by part A,

$$[\mathcal{G}_N : Q_N] \to [\mathcal{G}; Q], \tag{15}$$

and $w=f(z)$ maps the z-plane onto $\mathcal{G}$. Thus we can determine the structure of $\mathcal{G}$ from the structure of $\mathcal{G}_N$.

4. We next determine the structure of $\mathcal{G}_N$. First, $\mathcal{G}_N$ will have branch points over

$$a_{N,\nu} = f_N(b_\nu), \qquad 1 \leqq \nu \leqq n. \tag{16}$$

If the inequalities in (8) are all strict, these branch points are all of order one. Otherwise, some branch points coalesce but their total branch order is still n.

We next note that the zeros of $(1+\pi(z)/N)$, of which there are k, will be distinct for $N>N_0$. Let these zeros be

$$\beta_{N,\nu}, \qquad 1 \leqq \nu \leqq k, \tag{17}$$

and set

$$f_N(\beta_{N,\nu}) = \alpha_{N,\nu}, \qquad 1 \leqq \nu \leqq k. \tag{18}$$

Then $\mathcal{G}_N$ has k branch points of order N over the points $\alpha_{N,\nu}$. Also

$$\lim_{N\to\infty} \beta_{N,\nu} = \infty. \tag{19}$$

More precisely, considering the asymptotic behavior of $\pi(z)$, the roots $\beta_{N,\nu}$ occur in non-real conjugate pairs except for the following exceptions:

(20) $k=2p$, $c_k>0$; no exceptions,

(21) $k=2p+1$, $c_k>0$; one exception: $\beta_{N,2p+1}<0$,

(22) $k=2p+1$, $c_k<0$; one exception: $\beta_{N,2p+1}>0$,

(23) $k=2p+2$, $c_k<0$; two exceptions: $\beta_{N,2p+1}<0$, $\beta_{N,2p+2}>0$.

We shall assume that N is so large that any of these real roots lie outside the interval $[0, b_n]$ of the real axis. In all four cases there are p pairs of conjugate roots $\beta_{N,\nu}$ and we set:

$$\begin{cases} \mathfrak{I}\beta_{N,\nu} > 0, & 1 \leqq \nu \leqq p, \\ \mathfrak{I}\beta_{N,\nu} < 0, & p+1 \leqq \nu \leqq 2p, \end{cases} \tag{24}$$

and

$$\beta_{N,p+\nu} = \bar{\beta}_{N,\nu}, \qquad 1 \leqq \nu \leqq p. \tag{25}$$

The two branch points of $\mathcal{G}_N$ over $\alpha_{N,\nu}$ and $\alpha_{N,p+\nu}$ are symmetrically placed in $\mathcal{G}_N$; precisely, if γ: $w=w(t)$, $0 \leqq t \leqq 1$, is a curve in $\mathcal{G}_N$ from Q_N to the branch point over $\alpha_{N,\nu}$, then $\bar{\gamma}$: $w=\overline{w(t)}$, $0 \leqq t \leqq 1$, is a curve in $\mathcal{G}_N$ from Q_N to the branch point over $\alpha_{N,p+\nu}=\alpha_{N,\nu}$.

$\mathcal{G}_N$ is the image of the z-plane, but since $f_N(z)$ is a polynomial, we see that $\mathcal{G}_N$ has a deleted branch point of order $kN+n$ over ∞.

5. For the moment we assume that

$$\mathfrak{I}\alpha_{N,\nu} \neq 0, \qquad 1 \leqq \nu \leqq 2p. \tag{26}$$

With each branch point over a point $\alpha_{N,\nu}$, $1 \leqq \nu \leqq 2p$, we associate a ray $\Gamma_{N,\nu}$ on $\mathcal{W}$ from $\alpha_{N,\nu}$ to ∞ so that: $\Gamma_{N,\nu+p}$ and $\Gamma_{N,\nu}$ are conjugate, and each $\Gamma_{N,\nu}$ lies entirely in the half-plane $\mathfrak{I}w \gtrless 0$ containing $\alpha_{N,\nu}$. Also any two of these rays on $\mathcal{W}$ shall be disjoint unless the corresponding values α are equal, in which case the rays shall be identical. That this construction is possible is easily proved by induction on p, starting with the α of smallest modulus. For the one or two possible real values of α, as in (21, 22, 23), $\Gamma_{N,\nu}$ shall be one of the intervals of the real axis determined by α.

Now $\mathcal{G}_N$ is cut up by $k(N+1)$ rays, $N+1$ over each $\Gamma_{N,\nu}$, $1 \leqq \nu \leqq k$, issuing from the branch point of order N over $\alpha_{N,\nu}$. Since none of these branch points are in the curve, γ_N, on $\mathcal{G}_N$, corresponding to the interval $[0, b_n]$ of the real axis in $\mathcal{Z}$, those (none, one, or two) rays Γ on the real axis in $\mathcal{W}$ may be chosen so that none of the real rays on $\mathcal{G}_N$ intersect γ_N. These $k(N+1)$

rays are all disjoint on $\mathcal{G}_N$ except for the common end point of the group from any one branch point, and all end at the deleted branch point over ∞. It follows from these facts and the simple connectivity of $\mathcal{G}_N$ that $\mathcal{G}_N$ falls into $kN+1$ pieces. Since γ_N lies in a single piece, $\mathcal{H}_N$, and contains all the branch points remaining, the other kN pieces are all schlicht coverings of $\mathcal{W}$ with one or more of the rays Γ removed, for such a slit $\mathcal{W}$ is simply connected. Since $\mathcal{G}_N$ is a $(kN+n+1)$-fold covering of $\mathcal{W}$, $\mathcal{H}_N$ is an $(n+1)$-fold covering, and considering the nature of γ_N, it is easily proved that $\mathcal{H}_N$ may be obtained by removing certain symmetrically placed rays over some of the $\Gamma_{N,\nu}$ from a surface $\mathcal{H}_N' \in W_n$ (cf. §1). In case (20) these cuts occur in conjugate pairs, the number of pairs being between 1 and p. In case (21) there will be one cut along a part of the negative real axis of S_1 as well as q pairs of symmetric cuts, $0 \leqq q \leqq p$. Case (22) is similar: the single real cut lies in S_{n+1}, along the positive or negative end of the real axis according as n is even or odd. In case (23) there are two real cuts, one in S_1 and one in S_{n+1}.

It is clear now that the assumption (26) may be dropped, for the branch points of $\mathcal{G}_N$ may be altered slightly, if need be, to satisfy (26), $\mathcal{G}_N$ cut up, and then the various pieces, $\mathcal{H}_N$ etc., altered to put the branch points back in their proper places. The essential fact to note here is that N was chosen so large that no branch point of order N occurs on the curve γ_N. This fact is significant when $N \to \infty$ as we shall note in §11.

6. From this dissection it follows that the surfaces $\mathcal{G}_N$, for given n, k and $a_{N,\nu}$ are exactly those surfaces which are obtainable by the following construction: *we start with a base surface* $\mathcal{H}$ *of class* W_n, *with branch points over* $a_{N,\nu}$, $1 \leqq \nu \leqq n$, *and sheets* S_ν *as in* §1. $\mathcal{H}$ *is cut symmetrically along* q *disjoint rays*, $1 \leqq q \leqq k$, *which form a combination of one or more of the following possibilities: a single cut from* $\alpha_1 < 0$ *along the negative end of the real axis in* S_1, *a single cut from* α_2 *along the* $(-)^n$ *end of the real axis in* S_{n+1}, *and, pairs of symmetrically placed conjugate cuts in* $\mathcal{H}$. *None of these cuts contains any branch points of* $\mathcal{H}$. *Along each cut* N *copies of the* w*-plane with the corresponding cut are attached cyclically; a branch point of order* N *is thus formed over the finite end of each cut. The resulting surface,* $\mathcal{H}^{(1)}$, *has* q *branch points o* *order* N *as well as the branch points of* $\mathcal{H}$ *and a deleted branch point of order* $qN+n$ *over* ∞; $\mathcal{H}^{(1)}$ *is a* $(qN+n+1)$*-fold covering of the* w*-plane. This process is now repeated:* $\mathcal{H}^{(1)}$ *is cut symmetrically and new sheets are added, creating new branch points of order* N, *etc. The process terminates with* $\mathcal{H}^{(r)}$ *when* k *branch points of order* N *have been created.*

In brief, $\mathcal{G}_N$ *is obtained by attaching, symmetrically,* k *"extensions of order* N*" to a surface of class* W_n *one "base point" of each extension being over* ∞.

The need for repetition in this process is clear, for some of the extensions may be "hung" onto previous extensions rather than onto $\mathcal{H}$ directly.

7. That any surface $\mathcal{G}_N$ obtainable in this way is associated with a unique function $f_N(z)$ satisfying (13), (8), and (9) is easily proved. For $\mathcal{G}_N$ is simply connected and the addition of a single point over ∞ gives a closed surface. Thus $\mathcal{G}_N$ is associated with a polynomial, $f_N(z)$, of degree $Nk+n+1$; the obvious normalization, $f_N(0)=0\in S_1$ and $f'_N(0)=1$ clearly implies (13), (8), and (9).

8. The significance of (20, 21, 22, 23) is now clear: in case (20) the extensions occur in p conjugate pairs; in case (21) there is a self-conjugate extension hung onto S_1 plus p pairs of extensions; in case (22) there is a self-conjugate extension hung onto S_{n+1} plus p pairs of extensions; in case (23) there are two self-conjugate extensions, one hung onto S_1, one hung onto S_{n+1}, plus p pairs of extensions.

9. We now determine the structure of $\mathcal{G}$ from that of $\mathcal{G}_N$ and (15). From (14) and (16) we have,

$$a_{N,\nu} \to a_\nu = f(b_\nu), \qquad N \to \infty,\ 1 \leqq \nu \leqq n. \tag{27}$$

It may also be proved directly that

$$\alpha_{N,\nu} \to \alpha_\nu, \qquad N \to \infty,\ 1 \leqq \nu \leqq k, \tag{28}$$

where the values α_ν are the finite asymptotic values of $f(z)$ corresponding to the k sectors in Z in which $e^{\pi(z)}$ tends to zero. Namely, if (28) is valid then clearly the kernel $\mathcal{G}$ will consist of a surface of class W_n with k symmetrically placed logarithmic extensions added (described precisely in §10), whereas if (28) were not valid, (15) would be impossible.

10. *The surface* $\mathcal{G} \in W_n^k$ *associated with* (6, 7, 8, 9) *may be obtained as follows* (*cf.* §6): *we start with a base surface* $\mathcal{H} \in W_n$, *with branch points over* a_ν, $1 \leqq \nu \leqq n$, *and sheets* $S_1, \cdots, S_{n+1}$ *as in* §1. $\mathcal{H}$ *is cut symmetrically along* q *disjoint* (*in* $\mathcal{H}$) *rays*, $1 \leqq q \leqq k$, *each ray ending over* $w = \infty$. *Two infinite spirals of sheets are hung along each cut, one on either shore: that is, if* R *is the ray in question, running from* $w = \alpha$ *to* $w = \infty$, *these two spirals are the two components of the Riemann surface of* $\log(w - \alpha)$ *cut along a ray* R; *we say then that a logarithmic extension has been attached to* $\mathcal{H}$ *along* R. *The resulting surface has* q *finitely located logarithmic branch points and* q *logarithmic branch points over* ∞. *This process is iterated until the final surface* $\mathcal{G}$ *has* k *finitely located logarithmic branch points over* $\alpha_1, \alpha_2, \cdots, \alpha_k$; $\mathcal{G}$ *also has* k *logarithmic branch points over* ∞.

If $\mathcal{G} \in W_n^{k,+}$ *there is no self-conjugate extension attached to* S_{n+1}. *There is* (*is not*) *a self-conjugate extension attached to* S_1 *if* k *is odd* (*even*).

If $\mathcal{G} \in W_n^{k,-}$ *there is a self-conjugate extension attached to* S_{n+1}. *There is* (*is not*) *a self-conjugate extension attached to* S_1 *if* k *is even* (*odd*).

Clearly, because of the normalization of $f(z)$, *if there is a self-conjugate extension hung onto* S_1, *then the affix* α *of the logarithmic branch point satisfies* $\alpha < 0$. *That is*, $f(0) = 0 \in S_1$.

11. The restriction that the various cuts in $\mathcal{H}$ (or the cuts in any subsequent stage) be disjoint is important. Otherwise $\mathcal{G}$ might be disconnected. For example, if S_1 is cut long two conjugate rays from α, $0 < \alpha < a_1$, and two logarithmic extensions added, the result is two surfaces, not one.

12. The proof of the proposition of §10 is clear from (15), (27), the structure of $\mathcal{G}_N$ and the definition of kernel. The details of the proof are readily carried out after noting that: 1) relations of the form (28) hold for some subsequence N_i of the integers, and 2) the various sheets of $\mathcal{G}$, with fringes added, form an open covering of $\mathcal{G}^\delta$ and any compact part $\mathcal{A}$ of $\mathcal{G}^\delta$ is contained in a finite number of these fringed sheets.

That $\mathcal{G}$ actually has k finitely-affixed logarithmic branch points (in other words, that some of the branch points of order N do not "slide out" of the kernel) follows from the fact that $f(z)$, as is apparent by (6) and (9) has k distinct asymptotic tracts associated with finite asymptotic values.

13. Since $\mathcal{G}$ has a finite number of branch points, actual and logarithmic, $\mathcal{G}$ may be represented by a graph. The general theory of such representations was developed by Elfving [5]. The symmetry of $\mathcal{G}$ is of particular interest here, but this symmetry will in general not be apparent in such a representation, because of the various branch points both over and not over the real axis. However, the complex will be symmetrical whenever there exists a Jordan curve Γ on $\mathcal{W}$ which passes through the projections of all the branch points of $\mathcal{G}$ and which is symmetric in the real axis. If all the logarithmic branch points lie over the real axis, the real axis is such a curve.

Another simple case is the following: 1) none of the finitely-affixed logarithmic branch points lie over the real axis and 2) there is a single branch point of order n over $a_1 = a_2 \cdots = a_n$. Now the restriction 2) is not of great moment in obtaining an overall representation of $\mathcal{G}$. If 1) is satisfied, let $|\Im\alpha_k| > \epsilon > 0$ and let D be the rectangular domain on $\mathcal{W}$ with vertices at max $a_\nu + \epsilon \pm i\epsilon$ and min $a_\nu - \epsilon \pm i\epsilon$. Then the various components of $\mathcal{G}$ which lie over D are all schlicht except for one, $\mathcal{D}$, which is an $(n+1)$-fold covering of D and contains all the algebraic branch points of $\mathcal{G}$. Let $\mathcal{D}^*$ be the $(n+1)$-fold covering of D with a single branch point of order n over a, where a is real. If $\mathcal{G}$ is altered by replacing $\mathcal{D}$ by $\mathcal{D}^*$—in short, by collapsing all the algebraic branch points of $\mathcal{G}$ into a single branch point—then the new surface $\mathcal{G}^*$ may be represented by a symmetric graph. This alteration of $\mathcal{G}$ to $\mathcal{G}^*$ clearly has no effect on the arrangement of the logarithmic extensions in $\mathcal{G}$.

14. We have shown that any surface $\mathcal{G} \in W_n^k$ is obtainable by the process of §10. Conversely any surface $\mathcal{G}$ obtainable by this process is of class W_n^k as we now prove. This result is of course a corollary of Elfving's results [5], but the following proof is more in keeping with the methods of this paper.

Let $\mathcal{G}$ be a Riemann surface over $\mathcal{W}$, obtainable by the process of §10, with the branch points of the basic surface $\mathcal{H}$

over a_ν, $1 \leqq \nu \leqq n$, and with k finitely affixed logarithmic branch points. Let $S_1, S_2, \cdots, S_{n+1}$, be the sheets of $\mathcal{H}$. We wish to prove that there exists $f(z)$ satisfying (6, 7, 8, 9) and such that $w = f(z)$ maps the z-plane onto $\mathcal{G}$, with $f(0) = 0 \in S_1$.

Now $\mathcal{G}$ is the kernel of a sequence of surfaces $\mathcal{G}_N$ of the type described in §6. Namely, each logarithmic extension in $\mathcal{G}$ is replaced by an algebraic extension and the points of $\mathcal{G}_N$ over ∞ are deleted. In the present case the branch points of $\mathcal{G}_N$ and of $\mathcal{G}$ will have the same affixes, which is simpler than the situation of §4. This simplicity is offset by the dependence of $\pi(z)$ and b_ν on N in the equations corresponding to (13).

There are four cases to consider:

(A) $k = 2p$, $\mathcal{G}$ has no self-conjugate extensions,
(B) $k = 2p+1$, $\mathcal{G}$ has a self-conjugate extension hung into S_1,
(C) $k = 2p+1$, $\mathcal{G}$ has a self-conjugate extension hung into S_{n+1},
(D) $k = 2p+2$, $\mathcal{G}$ has two self-conjugate extensions, one hung into S_1 and one in S_{n+1}.

Now $\mathcal{G}_N$ is the image of the z-plane by a polynomial, $w = f_N(z)$, where

$$f_N(0) = 0 \in S_1, \qquad f'_N(0) = 1, \tag{29}$$

$$f'_N(z) = \left\{ \prod_{j=1}^{k} \left(1 - \frac{z}{\beta_{N,j}}\right) \right\}^N \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_{N,\nu}}\right), \tag{30}$$

$$0 < b_{N,1} \leqq b_{N,2} \leqq \cdots \leqq b_{N,n}, \tag{31}$$

$$\begin{cases} \beta_{N,j} = \rho_{N,j} e^{i\theta_{N,j}}, & 0 < \theta_{N,j} < \pi, \qquad 1 \leqq j \leqq p, \\ \beta_{N,p+j} = \rho_{N,j} e^{-i\theta_{N,j}}, & \qquad 1 \leqq j \leqq p, \end{cases} \tag{32}$$

and

$$\begin{cases} \beta_{N,2p+1} < 0 & \text{in cases } (B) \text{ and } (D), \\ \beta_{N,2p+1} > b_{N,n} > 0 & \text{in case } (C), \\ \beta_{N,2p+2} > b_{N,n} > 0 & \text{in case } (D). \end{cases} \tag{33}$$

15. If $\{f_N(z)\}$ is normal in the z-plane then it will follow from II Theorem 11 and Corollary α that $f_N(z) \to f(z)$ and that $w = f(z)$ maps the z-plane onto $\mathcal{G}$. We shall prove that $\{f_N\}$ is normal and that any limit function f is of the form (6, 7, 8, 9) and hence that $\mathcal{G} \in W_n^k$. More precisely, in case (A), (B), (C),

or (D), $\mathcal{G}$ is in $W_n^{2p,+}$, $W_n^{2p+1,+}$, $W_n^{2p+1,-}$, or $W_n^{2p+2,-}$ respectively. The proofs for the first three cases are included in the proof of case (*D*) by the following ruse: to obtain case (A) set $\beta_{N,2p+1} = \beta_{N,2p+2} = \infty$; to obtain (B) set $\beta_{N,2p+2} = \infty$; to obtain (C) set $\beta_{N,2p+1} = \infty$ and then replace $\beta_{N,2p+2}$ by $\beta_{N,2p+1}$. This procedure is justified since the following proof of case (D) involves only the reciprocals of the $\beta_{N,j}$.

16. Now $\mathcal{G}$, and hence each $\mathcal{G}_N$, contains a schlicht disc in S_1 of radius 4δ and center $w=0$. By I Theorem 1 each $f_N(z)$ is therefore schlicht, and $\{f_N\}$ is subuniformly bounded, in $|z| < \delta$; hence $\{f_N\}$ is normal in $|z| < \delta$. The families $\{f_N'\}$ and $\{\log f_N'\}$ are also normal in $|z| < \delta$.

Let $\{N_i\}$ be a subsequence of the integers such that

$$\lim_{i\to\infty} b_{N_i,\nu} = b_\nu, \qquad 1 \leqq \nu \leqq n, \tag{34}$$

$$\lim_{i\to\infty} \theta_{N_i,j} = \theta_j, \qquad 0 \leqq \theta_j \leqq \pi,\ 1 \leqq j \leqq p, \tag{35}$$

and

$$f_{N_i}(z) \to f(z) \quad \text{subuniformly in } |z| < \delta. \tag{36}$$

By virtue of (31) and since $f_N'(z) \neq 0$ in $|z| < \delta$,

$$0 < \delta \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_n \leqq \infty. \tag{37}$$

It is easy to prove directly that $b_n < \infty$, but this fact will be a consequence of the end result and is not needed now.

By (34),

$$\prod_{\nu=1}^{n}\left(1 - \frac{z}{b_{N_i,\nu}}\right) \to \prod_{\nu=1}^{n}\left(1 - \frac{z}{b_\nu}\right) \tag{38}$$

subuniformly in $|z| < \infty$ as $i \to \infty$. Set

$$g_N(z) = \prod_{j=1}^{k}\left(1 - \frac{z}{\beta_{N,j}}\right)^N. \tag{39}$$

By (36) and (38),

$$g_{N_i}(z) \to g(z) = f'(z) \Big/ \prod_{\nu=1}^{n}\left(1 - \frac{z}{b_\nu}\right) \tag{40}$$

subuniformly in $|z|<\delta$. Since $f'(z)\neq 0$ in $|z|<\delta$, $|\beta_{N,j}|\geq\delta$ and

$$\log g_N(z) = -\sum_{q=1}^{\infty} C_{N,q}\frac{z^q}{q}, \qquad |z|<\delta, \tag{41}$$

where

$$C_{N,q} = N\sum_{j=1}^{k}\frac{1}{\beta^q_{N,j}}. \tag{42}$$

By (32),

$$C_{N,q} = N\left\{\frac{1}{\beta^q_{N,2p+1}} + \frac{1}{\beta^q_{N,2p+2}} + 2\sum_{j=1}^{p}\frac{\cos q\theta_{N,j}}{\rho^q_{N,j}}\right\}. \tag{43}$$

By (40) the following finite limits exist:

$$C_{N_i,q}\to C_q, \qquad i\to\infty,\ 1\leq q. \tag{44}$$

We now use the Dirichlet "box" argument [8, Theorem 201]: for $\epsilon>0$ there exists an integer m such that

$$m\theta_j\equiv\epsilon_j \pmod{2\pi}, \qquad |\epsilon_j|\leq\epsilon, \qquad 1\leq j\leq p.$$

In particular, choosing $\epsilon<\pi/4$, there exists a positive integer m such that $\cos 2m\theta_j>0$, $1\leq j\leq p$. By (35),

$$\cos 2m\theta_{N_i,j}\geq\eta>0, \qquad 1\leq j\leq p,\ i>i_0. \tag{45}$$

Hence when $N=N_i$, $i>i_0$, and $q=2m$, all the terms in (43) are positive and from (44) and (45) it follows that

$$N_i\sum_{j=1}^{k}|\beta_{N_i,j}|^{-2m}=O(1), \qquad i\to\infty,$$

and a fortiori

$$\frac{1}{|\beta_{N_i,j}|}=O(N_i^{-1/2m}), \qquad i\to\infty,\ 1\leq j\leq k. \tag{46}$$

From (46) and (39) it follows that, for any $R>0$, $\log g_{N_i}(z)$ is holomorphic in $|z|\leq R$ for $i>i_0(R)$. Also, for $|z|\leq R$,

$$\log g_{N_i}(z) + \sum_{q=1}^{2m} C_{N_i,q} \frac{z^q}{q} = - \sum_{q=2m+1}^{\infty} C_{N_i,q} \frac{z^q}{q}$$

$$= O\left\{ N_i \sum_{q=2m+1}^{\infty} \frac{|z|^q}{q N_i^{q/2m}} \right\}$$

$$= O\left\{ N_i \sum_{q=2m+1}^{\infty} \left(\frac{R}{N_i^{1/2m}} \right)^q \right\}$$

$$= O\left(\frac{R^{2m+1}}{N_i^{1/2m} - R} \right) = o(1), \qquad i \to \infty.$$

Hence, by (44),

$$\log g_{N_i}(z) \to - \sum_{q=1}^{2m} \frac{C_q}{q} z^q, \qquad i \to \infty, \tag{47}$$

subuniformly in $|z| < \infty$. Comparing (30), (38), (39), and (47), and setting $-C_q/q = c_q$,

$$f'_{N_i}(z) \to f'(z) = e^{\pi(z)} \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_\nu}\right), \qquad i \to \infty, \tag{48}$$

subuniformly in $|z| < \infty$, and

$$\pi(z) = \sum_{q=1}^{2m} c_q z^q, \qquad c_q \text{ real.} \tag{49}$$

Now, as remarked in §15, it follows that $f_N(z) \to f(z)$ and $w = f(z)$ maps $|z| < \infty$ onto $\mathcal{G}$. From (48) and (49) together with (29) and (37) compared with (6, 7, 8, 9) it follows that $\mathcal{G} \in W_n^h$ for some $h \leqq 2m$. But we know that $\mathcal{G}$ has k finitely-affixed logarithmic branch points and hence it follows from §10 that $h = k$, i.e., $c_q = 0$ for $k < q \leqq 2m$ and $c_k \neq 0$. Since $\mathcal{G}$ has algebraic branch points of total order n, it follows from (48) that $b_n < \infty$, which completes (37). Combining this result with §10 we have

THEOREM 1. *$\mathcal{G} \in W_n^k$ if and only if $\mathcal{G}$ is obtainable by the construction process of §10. Also $\mathcal{G} \in W_n^{k,+}$, ($W_n^{k,-}$) if and only if there is not (is) a self-conjugate logarithmic extension attached to S_{n+1}.*

17. Definition of the class W. This class is just the class W_n of §1 with $n=\infty$. That is, $\mathcal{G}$, a Riemann surface over $\mathcal{W}$ is in W if and only if:

1° there exists a sequence of real numbers $\{a_\nu\}_1^\infty$ which satisfies

$$0 < a_1, \qquad a_{2\nu} \leqq a_{2\nu\pm 1}, \qquad 1 \leqq \nu, \tag{50}$$

and such that infinitely many of the inequalities (50) are strict, and

2° if S_1 is the w-plane cut along the interval $(a_1, +\infty)$ of the real axis and if S_ν, $1<\nu$, is the w-plane cut along the two intervals $(a_{\nu-1}, (-)^\nu\infty)$ and $(a_\nu, (-)^{\nu+1}\infty)$, then $\mathcal{G}$ consists of the sheets S_ν, $1\leqq\nu$, with the shores of the cuts identified as in §1: that is, S_ν and $S_{\nu+1}$ are joined "criss-cross" along their cuts issuing from a_ν, for $1\leqq\nu$.

If $\mathcal{G}\in W$ then $\mathcal{G}$ is open and simply connected and has algebraic branch points over $w=a_\nu$; if the inequalities (50) are all strict then there is a unique first order branch point associated with each a_ν (although various of the a_ν may be equal). The simplest example of a surface of class W is the Riemann surface, $\mathcal{G}_0$, for $z=(\arccos w)^2$, where $a_\nu=(-1)^{\nu+1}$. Another method of characterizing the general $\mathcal{G}\in W$ is: $\mathcal{G}$ may be derived from $\mathcal{G}_0$ by a continuous displacement of the branch points of $\mathcal{G}_0$ over the real axis, the branch point over a_1 remaining over the positive real axis during the displacement.

In [12, Theorems II and IIa] we proved the following Theorem.

THEOREM 2. *A Riemann surface $\mathcal{G}$ over $\mathcal{W}$ is of class W if and only if $\mathcal{G}$ is the image of the z-plane by an entire function, $w=f(z)$, satisfying:*

$$f'(z) = e^{-\delta z}\prod_{\nu=1}^{\infty}\left(1-\frac{z}{b_\nu}\right), \qquad f(0)=0\in S_1, \tag{51}$$

$$0 < b_1 \leqq b_2 \leqq \cdots \leqq b_n \leqq \cdots, \qquad \sum_{\nu=1}^{\infty}\frac{1}{b_\nu} < \infty, \tag{52}$$

and

$$\delta \geqq 0. \tag{53}$$

Theorem 2 is a simple special case of Theorem 3, the principal theorem of this chapter.

18. Definition of the classes $\mathbf{W}^k$. *A Riemann surface* $\mathcal{G}$ *over* $\mathcal{W}$ *is of class* $\mathbf{W}^k$, *k a non-negative integer, if and only if* $\mathcal{G}$ *consists of a Riemann surface* $\mathcal{H} \in \mathbf{W}$ *enlarged with k symmetrically arranged logarithmic extensions. That is,* $\mathcal{G}$ *is derived from* $\mathcal{H}$ *by the process of* §10.

That $\mathcal{H} \in \mathbf{W}$, rather than $\mathcal{H} \in \mathbf{W}_n$ as was the case in §10, obviously has only one effect: $\mathcal{G}$ *contains at most one self-conjugate extension*, namely, one determined by the interval $(-\infty, \alpha)$, $\alpha < 0$, of the real axis in $S_1 \subset \mathcal{H}$. This self-conjugate extension is present if and only if k is odd. The restriction $\alpha < 0$ is for convenience in normalization only: we shall consider maps $w = f(z)$ of the z-plane onto $\mathcal{G}$ with $f(0) = 0 \in S_1$ and $f(\bar{z}) = \overline{f(z)}$. The condition $\alpha < 0$ is not a real restriction on the variety of surfaces obtained, for clearly we must have $\alpha < a_1$ and the translation $w_1 = w - a_0$, $\alpha < a_0 < a_1$, will reduce the general case to the case $\alpha < 0$.

The reason for requiring infinitely many of the inequalities in (50) to be strict is now apparent: if for some n, $a_n \neq a_{n+1} = a_\nu$ for all $\nu > n$ then the corresponding surface constructed as in §17 is in $\mathbf{W}_n^{1,-}$ and the surface constructed in §18 is in $\mathbf{W}_n^{k+1,-}$. Hence the restriction on (50) amounts to making the classes $\mathbf{W}^k$ disjoint from the classes $\mathbf{W}_n^k$.

For $k = 0$, $\mathbf{W}^0 = \mathbf{W}$.

In [12] we considered the class $\mathbf{W}^1$ and obtained (Theorems I and Ia) the form of the uniformizing functions. Again this result is contained in Theorem 3 which exposes a one-one correspondence between the surfaces of each class $\mathbf{W}^k$ and an associated class of entire functions.

19. Let $\mathcal{G} \in \mathbf{W}^k$, $k \geqq 0$, be fixed. Now $\mathcal{G}$ is the kernel of a convergent sequence, $\{\mathcal{G}_n; Q_n\}$, of Riemann surfaces over $\mathcal{W}$, where $\mathcal{G}_n \in \mathbf{W}_n^{k,+}$, $n > n_0$. Namely, $\mathcal{G}_n$ is formed from $\mathcal{G}$ by dropping the sheets S_j, $j > n+1$, of $\mathcal{H}$ and removing the cut over $(a_{n+1}, (-)^n \infty)$ from S_{n+1}. Since k is finite, there exists an n_0 such that all logarithmic extensions in $\mathcal{G}$ are connected (directly or by a "chain" of logarithmic extensions) to the sheets S_ν, $1 \leqq \nu \leqq n_0 + 1$. Hence in dropping S_j, $j > n+1 > n_0 + 1$, no logarithmic extensions are dropped and the resulting surface $\mathcal{G}_n$ is in $W_n^{k,+}$. Let $Q(Q_n)$ be the point of $\mathcal{G}(\mathcal{G}_n)$ in S_1 over $w=0$. Then $[\mathcal{G}_n; Q_n]\rightarrow[\mathcal{G}; Q]$.

This proposition is most easily proved as follows. By the definition of $\mathcal{G}_n$ it is clear that $\mathcal{G}^\delta$ has the embeddability property 1° of II §22. Let $[\mathcal{K}^\delta; Q]$ be any kernel of $\{\mathcal{G}_n^\delta; Q_n\}$ and let $\mathcal{A}\subset\mathcal{K}^\delta$ be a subsurface such that $\mathcal{A}^-\subset\mathcal{K}^\delta$ is compact and $Q\in\mathcal{A}$. Then $[\mathcal{A}; Q]\subset[\mathcal{G}_n^\delta; Q_n]$, $n>n_0$, and hence, by the construction of $\mathcal{G}_n$, $[\mathcal{A}; Q]\subset[\mathcal{G}^\delta; Q]$. Let $\{\mathcal{A}_j; Q\}$ be an exhaustion of $\mathcal{K}^\delta$. Then $[\mathcal{A}_j; Q]\subset[\mathcal{G}^\delta, Q]$ for all j and hence $[\mathcal{K}^\delta; Q]\subset[\mathcal{G}^\delta; Q]$. By the maximal property of any $\mathcal{K}^\delta$ we then have $[\mathcal{K}^\delta; Q]=[\mathcal{G}^\delta; Q]$. Hence $[\mathcal{G}_n^\delta; Q_n]\rightarrow[\mathcal{G}^\delta; Q]$. Finally, the reinstatement of the algebraic branch points is trivial, and $[\mathcal{G}_n; Q_n]\rightarrow[\mathcal{G}; Q]$.

20. Let $w=f_n(z)$ map $|z|<\infty$ onto $\mathcal{G}_n$ with

$$f_n(0) = 0 \in S_1, \qquad f_n'(0) = 1. \tag{54}$$

Since $\mathcal{G}_n\in W_n^{k,+}$ we have from (6), (8), and (9):

$$f_n'(z) = e^{p_n(z)}\prod_{\nu=1}^{n}\left(1-\frac{z}{b_{n,\nu}}\right), \tag{55}$$

$$0 < b_{n,1} \leqq b_{n,2} \leqq \cdots \leqq b_{n,n}, \tag{56}$$

$$p_n(z) = c_{n,1}z + \cdots + c_{n,k}z^k, \quad c_{n,j} \text{ real}, \ c_{n,k} > 0 \tag{57}$$

Since each $\mathcal{G}_n$ contains a schlicht disc of radius 4δ and center 0 in S_1, it follows, as in §16, that the functions $f_n'(z)$ are subuniformly bounded in $|z|<\delta$ and also that the functions $\log f_n'(z)$, determined by $\log f_n'(0)=0$, are holomorphic and subuniformly bounded in $|z|<\delta$. In particular, let M be a constant such that

$$|\log f_n'(z)| \leqq M, \qquad n \geqq 1, \ |z| \leqq \delta/2. \tag{58}$$

Since $f_n'\neq 0$ in $|z|<\delta$,

$$\delta \leqq b_{n,1}. \tag{59}$$

Now in $|z|<b_{n,1}$,

$$\log f_n'(z) = \sum_{\mu=1}^{k} z^\mu \left\{ c_{n,\mu} - \frac{1}{\mu} \sum_{\nu=1}^{n} \frac{1}{b_{n,\nu}^\mu} \right\} - \sum_{\mu=k+1}^{\infty} \frac{z^\mu}{\mu} \sum_{\nu=1}^{n} \frac{1}{b_{n,\nu}^\mu}. \tag{60}$$

By (58) and Cauchy's estimate,

$$\left| c_{n,\mu} - \frac{1}{\mu} \sum_{\nu=1}^{n} \frac{1}{b_{n,\nu}^\mu} \right| \leqq M \left(\frac{2}{\delta} \right)^\mu = M_\mu, \quad 1 \leqq \mu \leqq k, \tag{61}$$

and

$$\frac{1}{\mu} \sum_{\nu=1}^{n} \frac{1}{b_{n,\nu}^\mu} \leqq M \left(\frac{2}{\delta} \right)^\mu = M_\mu, \qquad k < \mu. \tag{62}$$

For $\sigma \leqq n$, $k < \mu$, we have from (62) and (56),

$$\frac{\sigma}{b_{n,\sigma}^\mu} \leqq \sum_{\nu=1}^{\sigma} \frac{1}{b_{n,\nu}^\mu} \leqq \mu M_\mu,$$

or, setting $\mu = k+1$,

$$b_{n,\sigma} \geqq A \sigma^{1/(k+1)}, \qquad n \geqq 1,\ 1 \leqq \sigma \leqq n, \tag{63}$$

where $A > 0$ is a constant.

For any fixed m, and $n > m$, the Taylor series for

$$F_{n,m}(z) = \log \left\{ f_n'(z) \Big/ \prod_{\nu=1}^{m-1} \left(1 - \frac{z}{b_{n,\nu}} \right) \right\}$$

in $|z| < b_{n,m}$ differs from the Taylor series in (60) only in the lower limits on the two summations over ν: these limits will be m instead of 1. Then by (61), for $|z| < Am^{1/(k+1)} \leqq b_{n,m}$, we have

$$\begin{aligned} |F_{n,m}(z)| \leqq & \sum_{\mu=1}^{k} |z|^\mu \left\{ M_\mu + \frac{1}{\mu} \sum_{\nu=1}^{m-1} \frac{1}{b_{n,\nu}} \right\} \\ & + \sum_{\mu=k+1}^{\infty} \frac{|z|^\mu}{\mu} \sum_{\nu=m}^{n} \frac{1}{b_{n,\nu}^\mu}. \end{aligned} \tag{64}$$

Now by (63),

$$M_\mu + \frac{1}{\mu} \sum_{\nu=1}^{m-1} \frac{1}{b_{n,\nu}^\mu} < B, \qquad 1 \leqq \mu \leqq k, \tag{65}$$

where B is a constant depending only on m. Also by (63), for $\mu > k+1$,

$$\begin{aligned}\frac{1}{\mu}\sum_{\nu=m}^{n}\frac{1}{b_{n,\nu}^{\mu}} &\leqq \frac{1}{\mu A^{\mu}}\sum_{\nu=m}^{n}\nu^{-\mu/(k+1)}\\ &< \frac{m^{-\mu/(k+1)}}{\mu A^{\mu}}+\frac{1}{\mu A^{\mu}}\int_{m}^{\infty}x^{-\mu/(k+1)}dx \qquad (66)\\ &< (m+1)\left(\frac{1}{Am^{1/(k+1)}}\right)^{\mu}.\end{aligned}$$

Using (65) and (66), and (62) for $\mu=k+1$, in (64),

$$\begin{aligned}|F_{n,m}(z)| &< \sum_{\mu=1}^{k}B|z|^{\mu}+M_{k+1}|z|^{k+1}\\ &\quad+(m+1)\sum_{\mu=k+2}^{\infty}\left(\frac{|z|}{Am^{1/(k+1)}}\right)^{\mu}.\end{aligned}$$

Now this estimate is valid and finite for $|z|<Am^{1/(k+1)}$ and is independent of n. Thus the functions $f_n'(z)/\prod_{\nu=1}^{m-1}(1-z/b_{n,\nu})$ are uniformly bounded in $|z|\leqq\frac{1}{2}Am^{1/(k+1)}$. By (63) the functions $\prod_{\nu=1}^{m-1}(1-z/b_{n,\nu})$ are also uniformly bounded in this same region and hence, for any fixed m, the functions $f_n'(z)$ are uniformly bounded in $|z|\leqq\frac{1}{2}Am^{1/(k+1)}$. Since m is arbitrary, *the family* $\{f_n'\}$ *is subuniformly bounded, and hence normal, in* $|z|<\infty$. *Finally,* $f_n(0)=0$ *and therefore* $\{f_n\}$ *is a subuniformly bounded normal family in* $|z|<\infty$.

21. We can now apply II Theorem 11 with Corollary α to the sequence of maps from $\mathcal{J}_n=\{|z|<\infty\}$ onto $\mathcal{G}_n$ and conclude that

$$f_n(z)\rightarrow f(z), \qquad n\rightarrow\infty, \tag{67}$$

subuniformly in $|z|<\infty$ and that $w=f(z)$ maps $|z|<\infty$ one-one onto $\mathcal{G}$ with $f(0)=0\in S_1$ and $f'(0)=1$. Also $f_n'(z)\rightarrow f'(z)$.

We consider next the form of $f'(z)$; for this purpose it is sufficient to work in $|z|<\delta$. Consulting the remarks between (57) and (58), we obtain

$$\log f_n'(z)\rightarrow\log f'(z) \tag{68}$$

subuniformly in $|z|<\delta$. If

$$\log f'(z)=\sum_{\mu=1}^{\infty}d_{\mu}z^{\mu}, \qquad |z|<\delta. \tag{69}$$

then by (60)

$$d_\mu = -\lim_{n\to\infty}\frac{1}{\mu}\sum_{\nu=1}^{n}\frac{1}{b_{n,\nu}^{\mu}}, \qquad \mu > k. \tag{70}$$

Now $f'(z)\not\equiv 0$ and by Hurwitz' theorem the zeros of $f_n'(z)$ tend to the zeros of $f'(z)$, which are real and positive and correspond to the algebraic branch points of $\mathcal{G}$. If the zeros of $f'(z)$ are b_ν,

$$0 < \delta \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_\nu \leqq b_{\nu+1} \leqq \cdots \tag{71}$$

where an mth order zero of $f'(z)$ is represented by m consecutive equal b's, then

$$\lim_{n\to\infty} b_{n,\nu} = b_\nu, \qquad 1 \leqq \nu. \tag{72}$$

Also of course, since corresponding branch points of $\mathcal{G}_n$ and $\mathcal{G}$ have the same order, $b_\nu = b_{\nu+1}$ if and only if $b_{n,\nu} = b_{n,\nu+1}$ for all $n > \nu$. As a matter of fact, it may be shown by means of the known extremal domains associated with I Theorem 1 that $b_{n,\nu} \uparrow b_\nu$ as $n \uparrow \infty$. But this fact is not required in the ensuing development. From (63) we have

$$b_\nu \geqq A\nu^{1/(k+1)}, \qquad A = \text{constant} > 0,\ 1 \leqq \nu. \tag{73}$$

Therefore

$$\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^\mu} < \infty, \qquad \mu > k+1,$$

and by (70), for any fixed N, and $\mu > k+1$,

$$\left| d_\mu + \frac{1}{\mu}\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^\mu}\right| = \left|\lim_{n\to\infty}\frac{1}{\mu}\sum_{\nu=1}^{n}\left(\frac{1}{b_\nu^\mu} - \frac{1}{b_{n,\nu}^\mu}\right)\right|$$

$$\leqq \limsup_{n\to\infty}\left|\frac{1}{\mu}\sum_{\nu=1}^{N}\left(\frac{1}{b_\nu^\mu} - \frac{1}{b_{n,\nu}^\mu}\right)\right| + \frac{1}{\mu}\sum_{\nu=N+1}^{\infty}\frac{2}{A^\mu \nu^{\mu/(k+1)}},$$

where the last sum is derived with the aid of (63) and (73). The limsup term vanishes, by (72), and hence

$$\left| d_\mu + \frac{1}{\mu}\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^\mu}\right| \leqq \frac{2}{\mu A^\mu}\sum_{\nu=N+1}^{\infty}\nu^{-\mu/(k+1)} = O(N^{1-\mu/(k+1)}) = o(1)$$

as $N \to \infty$. Therefore

$$d_\mu = -\frac{1}{\mu}\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^\mu}, \qquad \mu > k+1. \tag{74}$$

For $\mu = k+1$, we have from (70), for any fixed N,

$$\begin{aligned} -d_{k+1} &= \lim_{n\to\infty}\frac{1}{k+1}\sum_{\nu=1}^{n}\frac{1}{b_{n,\nu}^{k+1}} \\ &\geqq \lim_{n\to\infty}\frac{1}{k+1}\sum_{\nu=1}^{N}\frac{1}{b_{n,\nu}^{k+1}} = \frac{1}{k+1}\sum_{\nu=1}^{N}\frac{1}{b_\nu^{k+1}}. \end{aligned}$$

Therefore

$$\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^{k+1}} \leqq -(k+1)d_{k+1} < \infty,$$

and

$$d_{k+1} = -\frac{1}{k+1}\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^{k+1}} + c_{k+1}, \tag{75}$$

where

$$c_{k+1} \leqq 0. \tag{76}$$

Combining (69), (74), and (75),

$$\begin{aligned} \log f'(z) &= \sum_{\mu=1}^{k} d_\mu z^\mu + c_{k+1}z^{k+1} + \sum_{\mu=k+1}^{\infty} z^\mu\left(-\frac{1}{\mu}\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^\mu}\right) \\ &= \sum_{\mu=1}^{k} d_\mu z^\mu + c_{k+1}z^{k+1} + \log\prod_{\nu=1}^{\infty} E\left(\frac{z}{b_\nu}, k\right), \end{aligned} \tag{77}$$

where, as usual,

$$E(u, k) \equiv (1-u)\exp\sum_{j=1}^{k}\frac{u^j}{j}. \tag{78}$$

If q is the integer determined by

$$\sum_{\nu=1}^{\infty}\frac{1}{b_\nu^q} = \infty, \qquad \sum_{\nu=1}^{\infty}\frac{1}{b_\nu^{q+1}} < \infty, \tag{79}$$

then $0 \leqq q \leqq k$ and (77) may be put in the form

$$\log f'(z) = \sum_{\mu=1}^{k+1} c_\mu z^\mu + \log \prod_{\nu=1}^{\infty} E\left(\frac{z}{b_\nu}, q\right), \qquad |z| < \delta, \tag{80}$$

where

$$c_\mu = d_\mu, \qquad \text{for } 1 \leqq \mu \leqq q, \text{ and}$$

$$c_\mu = d_\mu + \frac{1}{\mu} \sum_{\nu=1}^{\infty} \frac{1}{b_\nu^\mu}, \qquad \text{for } q < \mu \leqq k.$$

22. Collecting the results of §§19–21, and in particular (76), (79), and (80), we have proved: *if* $\mathcal{G} \in W^k$, $k \geqq 0$, *then* $\mathcal{G}$ *is parabolic and there exists a unique entire function* $f(z)$ *such that* $w = f(z)$ *maps* $|z| < \infty$ *onto* $\mathcal{G}$ *with* $f(0) = 0 \in S_1$ *and* $f'(0) = 1$. *Furthermore* $f'(z)$ *is of the form*

$$f'(z) = e^{\pi(z)} \prod_{\nu=1}^{\infty} E\left(\frac{z}{b_\nu}, q\right), \tag{81}$$

where

$$0 < b_\nu \leqq b_{\nu+1}, \qquad 1 \leqq \nu, \tag{82}$$

q *is determined by* (79), $0 \leqq q \leqq k$, *and*

$$\pi(z) = \sum_{\mu=1}^{k+1} c_\mu z^\mu, \qquad c_\mu \textit{ real}, \ c_{k+1} \leqq 0. \tag{83}$$

As yet we have insufficient knowledge to determine the precise degree of $\pi(z)$. This information will be a consequence of the converse problem which we investigate next.

23. Let $f(z)$ be an entire function such that

$$\begin{cases} f(0) = 0, \\ f'(z) \textit{ is of the form } (81), \textit{ and} \\ (79) \textit{ and } (82) \textit{ are satisfied.} \end{cases} \tag{84}$$

In addition, let $\sigma \geqq 0$ and let

$$\pi(z) = \sum_{\mu=1}^{\sigma} c_\mu z^\mu, \qquad c_\mu \text{ real}, \ c_\sigma \neq 0; \tag{85}$$

for $\sigma=0$ we interpret (85) as $\pi(z)\equiv 0$. Let

$$\gamma = \max(\sigma, q) = \text{genus of } f'(z). \tag{86}$$

We wish to determine the structure of the Riemann surface $\mathcal{G}$ over $\mathcal{W}$ for the inverse function $f^{-1}(w)$. Define $f_n(z)$ by

$$f_n'(z) = e^{\pi(z)} \prod_{\nu=1}^{n} E\left(\frac{z}{b_\nu}, q\right), \qquad n \geqq 1, \tag{87}$$

and $f_n(0)=0$. Then $f_n(z)\to f(z)$ subuniformly in $|z|<\infty$. Also (87) may be rearranged to

$$f_n'(z) = e^{p_n(z)} \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_\nu}\right), \tag{88}$$

where

$$p_n(z) = \pi(z) + \sum_{\mu=1}^{q} \frac{z^\mu}{\mu} \sum_{\nu=1}^{n} \frac{1}{b_\nu^\mu} = \sum_{\mu=1}^{\gamma} d_{n,\mu} z^\mu.$$

Comparing these equations with (6), (7), (8), and (9) we see that, for n large enough, $w=f_n(z)$ maps $|z|<\infty$ onto a Riemann surface $\mathcal{G}_n \in \mathbf{W}_n^\gamma$, where γ is defined by (86). That $d_{n,\gamma}$ is not zero for $n>n_0$ may be proved as follows. If $\sigma>q$, then $d_{n,\gamma}=c_\sigma\neq 0$, by (85). If $q\geqq\sigma$, then $d_{n,\gamma}=\delta_{\sigma,q}c_\sigma+1/q\sum_{\nu=1}^{n} b_\nu^{-q}$, where $\delta_{\sigma,q}$ is the Kronecker δ; then by virtue of (79), $d_{n,\gamma}\to+\infty$ as $n\to\infty$. Thus we have, for $n>n_0$,

$$\mathcal{G}_n \in \begin{cases} \mathbf{W}_n^{\gamma,+} & \textit{for } q \geqq \sigma \textit{ or } q<\sigma,\ c_\sigma>0, \\ \mathbf{W}_n^{\gamma,-} & \textit{for } q<\sigma,\ c_\sigma<0. \end{cases} \tag{89}$$

Since $f'(0)=1$ and $f_n(z)\to f(z)$, each $f_n(z)$ is schlicht in a fixed disc $|z|<\delta$ and, by I Theorem 1, each $\mathcal{G}_n$ contains a schlicht disc with center $Q_n\in S_1$ over $w=0$ and radius $\delta/4$. Hence by II Theorem 11, with Corollary α, $[\mathcal{G}_n; Q_n]\to[\mathcal{G}; Q]$ and $w=f(z)$ maps $|z|<\infty$ one-one onto $\mathcal{G}$.

By a comparison of Theorem 1 with §§17 and 18 it is clear that $\mathcal{G}\in\mathbf{W}^k$, for some integer $k\geqq 0$, and that $Q\in S_1$. Now $\mathcal{G}$ can have at most as many finitely-affixed logarithmic branch points as $\mathcal{G}_n$, because $\mathcal{G}_n\to\mathcal{G}$. Since $\mathcal{G}_n$ has exactly γ such branch points we see that $k \leqq \gamma$. The question is then: how many of these γ logarithmic branch points can "disappear" as $n \to \infty$? Such a "disappearance" occurs whenever the affix in question tends to ∞ or whenever the corresponding logarithmic extension is attached "farther and farther out" on the surface, relative to Q_n. For example, if the extension in question is attached to the mth sheet of the basic surface from which $\mathcal{G}_n$ is constructed and if $m \to \infty$ as $n \to \infty$, then no corresponding extension will appear in the kernel $\mathcal{G}$. In particular, if $\mathcal{G}_n \in W_n^{\gamma,-}$ then the self-conjugate extension attached to S_{n+1} in $\mathcal{G}_n$ will "disappear" in the limit and in this case $k \leqq \gamma - 1$. By virtue of (89) this case occurs whenever $q < \sigma$ and $c_\sigma < 0$.

24. To restate the results of §23, *if* $\mathcal{G}$ *is the image of* $|z| < \infty$ *by* $w = f(z)$, *where* $f(z)$ *is defined by* (84), (85), *and* (86), *then* $\mathcal{G} \in W^k$, $f(0) = 0 \in S_1$, *and* k *satisfies*

$$\begin{cases} k \leqq \gamma, & \text{for } q \geqq \sigma \text{ or } q < \sigma,\ c_\sigma > 0, \\ k \leqq \gamma - 1, & \text{for } q < \sigma,\ c_\sigma < 0. \end{cases} \tag{90}$$

But now that we know $\mathcal{G} \in W^k$ it follows from §22 that $q \leqq k$ and $\sigma \leqq k+1$. Together with (90) these inequalities give: if $\gamma = q \geqq \sigma$, then $k \leqq \gamma = q \leqq k$ and hence $k = \gamma$; if $q < \sigma$, $c_\sigma > 0$, then, by (83), $\sigma < k+1$ and hence, by (90), $\gamma = \sigma \leqq k \leqq \gamma$ and therefore $k = \gamma$; finally, if $q < \sigma$, $c_\sigma < 0$, then $\gamma = \sigma \leqq k+1 \leqq \gamma$ and hence $k = \gamma - 1$. Thus the two inequalities in (90) may be replaced by $k = \gamma$ and $k = \gamma - 1$.

Finally, combining §§22 and 24, we have proved the fundamental theorem of this chapter [13]:

THEOREM 3. *Let the classes,* W^k, *of Riemann surfaces be defined as in* §§17 *and* 18. *Then* $\mathcal{G} \in W^k$ *if and only if* $\mathcal{G}$ *is the one-one image of* $|z| < \infty$ *by an entire function* $w = f(z)$ *satisfying the following six conditions:*

$$f(0) = 0 \in S_1, \qquad f'(0) = 1, \tag{91}$$

$$f'(z) = e^{\pi(z)} \prod_{\nu=1}^{\infty} E\left(\frac{z}{b_\nu}, q\right), \qquad E \text{ as in } (78), \tag{92}$$

$$0 < b_\nu \leqq b_{\nu+1}, \qquad 1 \leqq \nu \tag{93}$$

$$\sum_{\nu=1}^{\infty} \frac{1}{b_\nu^q} = \infty, \qquad \sum_{\nu=1}^{\infty} \frac{1}{b_\nu^{q+1}} < \infty, \tag{94}$$

$$\pi(z) = \sum_{\mu=1}^{\sigma} c_\mu z^\mu, \qquad c_\mu \text{ real}, \ c_\sigma \neq 0, \tag{95}$$

and, if $\gamma = \max(\sigma, q)$ *is the genus of* $f'(z)$,

$$\begin{cases} k = \gamma, & \text{for } q \geqq \sigma \text{ or } q < \sigma, \ c_\sigma > 0, \\ k = \gamma - 1, & \text{for } q < \sigma, \ c_\sigma < 0. \end{cases} \tag{96}$$

If $\sigma = 0$ *then* (95) *is to be interpreted as* $\pi(z) \equiv 0$.

For any fixed $k \geqq 0$ *the correspondence between* W^k *and all entire functions satisfying* (91) *through* (96), (*with the given* k), *is one-one.*

25. One way of summarizing Theorems 1 and 3 is that we have established a one-one correspondence between the union of all the classes W_n^k and W^k, $n \geqq 0$, $k \geqq 0$, and the set of all so-called real entire functions, $f'(z)$, of finite order with all positive zeros, and such that $f'(0) = 1$.

If we set $k=0$ or 1 in Theorem 3 we obtain Theorem 2 or [12, Theorems I and Ia].

26. The structure of $\mathcal{G}$ as related to the form of $f(z)$ by Theorem 3 enables us to determine the asymptotic behavior of $f(z)$ and $f'(z)$. The procedure is the following: by Iversen's theorem (see I §9) the tracts of determination of $f(z)$ correspond to the singularities of $\mathcal{G}$. Then by the form of $f(z)$, as given by Theorem 3, we can locate these tracts of determination fairly accurately in the z-plane and also show that $f'(z)$ has essentially the same tracts of determination.

As we shall see, the sectors corresponding to the tracts of determination are not always separated by straight rays, and in such cases the use of $\mathcal{G}$ leads to more complete results on the asymptotic behavior of f and f' than we have been able to derive directly from (91) through (96). The following theorem summarizes some of the results derived in the rest of this chapter.

THEOREM 4. *Let* $f(z)$ *be given by* (91) *through* (96), *with* $k > 0$. *Then there exist* $2k$ *curves* $L_1, L_2, \cdots, L_{2k}$, *running from* $z=0$ *to*

$z=\infty$, which are disjoint except for the points 0 *and* ∞, *and which have the following properties:*

1° *L_1 lies in $\Im z \geqq 0$, $L_1, L_2, \cdots, L_{2k}$ progress cyclically in the positive sense about $z=0$, and L_{2k+1-j} is the reflection of L_j in the real axis, $1 \leqq j \leqq 2k$.*

2° *These curves L_j have a natural connection with $f(z)$ in the following sense: there exists a function $\psi(\zeta)$ which maps the ζ-plane, cut along a segment (b, ∞), $b>0$, of the real axis, one-one and conformally onto a domain D_z of the z-plane, such that: $\psi(0)=0$, $\psi'(0)=1$, $\psi(\zeta)$ real for ζ real, L_j is the image of* $\arg \zeta=(2j-1)\pi/2k$, *and in D_z: $f(z)=f_n(\zeta)$, where $f_n(\zeta)$ is a function of the form* (54, 55, 56, 57) *with z replaced by ζ.*

3° *Let T_j, $1 \leqq j \leqq 2k-1$, denote that sector bounded by L_j and L_{j+1} which contains no curve L_i (does not contain the positive real axis, if $k=1$). Then $f(z)$ and $f'(z)$ have the asymptotic value ∞ in T_j for $j=2, 4, \cdots, 2k-2$; $f(z)$ has a finite asymptotic value and $f'(z)$ has the asymptotic value* 0 *in T_j for $j=1, 3, \cdots, 2k-1$. These statements are to be interpreted according to the definition of* I §10.

4° **Case** *α: $c_\sigma>0$, $q<\sigma=k$. In this case,* $\arg z \to (2j-1)\pi/2k$ *as $z \to \infty$ on L_j, $1 \leqq j \leqq 2k$. Also the sectors T_j, $1 \leqq j \leqq 2k-1$, represent all but one of the asymptotic tracts of $f(z)$ and $f'(z)$. This one remaining tract, for either f or f', may be represented by a telescoping sequence of domains in* $|\arg z| < (\pi+\epsilon)/2k$ *in which the function in question tends to ∞. These domains are not simply connected and* I §10 *does not apply. This last tract of determination is associated with a singularity of $\mathcal{G}$ over $w=\infty$ which may be described as "a logarithmic branch point complicated with a condensation of algebraic branch points." The other asymptotic values of $f(z)$ are associated with logarithmic branch points of $\mathcal{G}$.*

5° **Case** *β: $c_\sigma<0$, $q<\sigma$, $k=\sigma-1$. In this case* $\arg z \to (2j+1)\pi/2\sigma$ *as $z \to \infty$ on L_j, $1 \leqq j \leqq 2k$. Also, the sectors T_j, $1 \leqq j \leqq 2k-1$, represent all but three of the asymptotic tracts of f and f'. These three further tracts are associated with the following asymptotic values: f and f' both have the asymptotic value ∞ in the sectors U_1 and $U_{2\sigma-1}$ and finite asymptotic values in U_0, where U_j is defined by* (137). *In this case the algebraic branch points of $\mathcal{G}$ condense to form a singularity of $\mathcal{G}$ over a finite point of $\mathcal{W}$, this singularity being* *associated with the asymptotic values in* U_0. *All the other asymptotic values are associated with logarithmic branch points of* $\mathcal{G}$.

6° **Case** γ: $\sigma \leqq q = k$. *In this case*

$$T_j \supset V_j(\epsilon) \cap \{ |z| > M(\epsilon) \}, \qquad \epsilon > 0,\ 1 \leqq j \leqq 2k-1,$$

where $V_j(\epsilon)$ *is defined by* (144) *and the last sentence of* §42. *Furthermore, this is the best that can be said as far as the oscillation of* arg z *on* L_j *is concerned; a detailed formulation of this fact will be found in* §45.

27. As a preliminary we consider the classes $W_n^{k,+}$, $k>0$, and the associated functions according to Theorem 1. The situation here is simple, but the results are useful in deriving the asymptotic properties of the functions occurring in Theorem 3, as is clear from 2° in Theorem 4. Hence we consider these simpler classes in some detail.

For $\mathcal{G} \in W_n^k$, $k>0$, $\mathcal{G}$ has no singularities except for k finitely-affixed logarithmic branch points and k logarithmic branch points over $w = \infty$, (see §10). By Iversen's theorem (I §9) these $2k$ logarithmic branch points correspond to the $2k$ distinct tracts of determination of $f(\zeta)$. Here f is the function associated with $\mathcal{G}$ by Theorem 1. We shall replace z by ζ in (6), (7), (8), (9), etc., since we wish to reserve z for later purposes.

Let Σ_j denote the sector

$$\Sigma_j: \frac{(2j-1)\pi}{2k} < \arg \zeta < \frac{(2j+1)\pi}{2k}, \quad 0 \leqq j \leqq 2k-1. \tag{97}$$

For $0 < \epsilon < \pi/2k$, let $\Sigma_j(\epsilon)$ denote the closed sector

$$\Sigma_j(\epsilon): \frac{(2j-1)\pi}{2k} + \epsilon \leqq \arg \zeta \leqq \frac{(2j+1)\pi}{2k} - \epsilon, \quad 0 \leqq j \leqq 2k-1. \tag{98}$$

Since $\mathcal{G} \in W_n^{k,+}$, $c_k > 0$ and by (6) and (9) we have:

$$|f'(\zeta)| < A_1 \exp(-\kappa_1 |\zeta|^k), \quad \zeta \in \Sigma_j(\epsilon),\ j = 1, 3, 5, \cdots, 2k-1, \tag{99}$$

and

$$|f'(\zeta)| > A_2 \exp(\kappa_2 |\zeta|^k), \quad \zeta \in \Sigma_j(\epsilon),\ j = 0, 2, \cdots, 2k-2, \tag{100}$$

where A_1, A_2, κ_1, κ_2 are positive constants depending on $\epsilon > 0$. Hence for j odd (even) $f'(\zeta)$ has the asymptotic value $0(\infty)$ in Σ_j, in the sense of I §10. These $2k$ sectors correspond to $2k$ distinct tracts of determination of $f'(\zeta)$, and $f'(\zeta)$ has no other tracts of determination. This last statement may be proved in various ways. It follows from Elfving's results [5] that the Riemann surface $\mathcal{H}$ for the inverse of $f'(\zeta)$ possesses $2k$ logarithmic branch points and no other singularities. Hence $f'(\zeta)$ has exactly $2k$ distinct tracts of determination. Alternatively, the structure of $\mathcal{H}$ may be deduced by methods parallel to those used in the proof of Theorem 1, although $\mathcal{H}$ does not in general possess the proper sort of symmetry to belong to any class W_n^k. The symmetry we have assumed became essential in the transition from W_n^k to W^k which was accomplished by letting $n \to \infty$.

Another method of proving that $f'(\zeta)$ has no further tracts of determination is based on theorems of Lindelöf. We shall use this method in §37.

The asymptotic values of $f(\zeta)$ are associated with these same sectors Σ_j. From (99) we have

(101) $f(\zeta) \to \alpha_j \neq \infty$ *uniformly in* $\Sigma_j(\epsilon)$, $j = 1, 3, \cdots, 2k-1$,

where the α_j are the affixes of the finitely located logarithmic branch points of $\mathcal{G}$. By the symmetry of $\mathcal{G}$, $\alpha_{2k-j} = \bar{\alpha}_j$ and the two tracts of determination associated with Σ_j and Σ_{2k-j}, j odd, correspond to a conjugate pair of logarithmic branch points. More precisely, (101) may be replaced by

$$f(\zeta) = \alpha_j + \frac{f'(\zeta)}{\pi'(\zeta)}\,(1 + O(|\zeta|^{-k})), \tag{102}$$

which holds uniformly as $\zeta \to \infty$ *in* $\Sigma_j(\epsilon)$, $j = 1, 3, \cdots, 2k-1$. The usual procedure of integration by parts will prove (102) as well as

$$f(\zeta) = \frac{f'(\zeta)}{\pi'(\zeta)}\,(1 + O(|\zeta|^{-k})), \tag{103}$$

which holds uniformly as $\zeta \to \infty$ *in* $\Sigma_j(\epsilon)$, $j = 0, 2, \cdots, 2k-2$. From (103) and (100) we derive

$$f(\zeta) \to \infty, \text{ uniformly as } \zeta \to \infty \text{ in } \Sigma_j(\epsilon), \qquad j = 0, 2, \cdots, 2k-2. \tag{104}$$

Hence, by (101) and (104), $f(\zeta)$ possesses $2k$ distinct tracts of determination corresponding to the $2k$ sectors Σ_j and by Iversen's theorem $f(\zeta)$ has no further tracts of determination.

28. From the analytic aspects of $f(\zeta)$ as given by (6) through (9) it is natural to think of the tracts of f or f' as being represented by the sectors Σ_j. From the geometric side and in view of the definition of a tract of determination a more natural decomposition of the ζ-plane may be obtained as follows.

Let $M>0$ be a constant such that all of the finitely affixed algebraic and logarithmic branch points of $\mathcal{G}$ lie over $|w|<M$. Then the part of $\mathcal{G}$ which lies over $|w|>M$ consists of k components, each of which is a logarithmic neighborhood of ∞, i.e., that part of the Riemann surface for log w which lies over $|w|>M$. These k neighborhoods are bounded by k disjoint curves on $\mathcal{G}$. By the choice of M there is only one component of $\mathcal{G}$ over $|w|<M$. Hence these boundary curves on $\mathcal{G}$ correspond to k disjoint analytic curves γ_{2j}, $0 \leqq j \leqq k-1$ in the ζ-plane, with the properties: 1) each γ_{2j} is a simple curve along which $\zeta \to \infty$ when γ_{2j} is traversed in either sense, 2) γ_{2j} splits the ζ-plane into two domains, Δ_{2j} and E_{2j} such that

$$|f(\zeta)| > M, \qquad \zeta \in \Delta_{2j}; \qquad \zeta = 0 \in E_{2j}, \tag{105}$$

and such that $w=f(\zeta)$ maps Δ_{2j} onto one of the logarithmic neighborhoods of ∞ in $\mathcal{G}$. The domains Δ_{2j} are disjoint and the remaining domain, $\Delta^* = \bigcap E_{2j}$, which is characterized by

$$|f(\zeta)| < M, \qquad \zeta \in \Delta^*, \tag{106}$$

is bounded by all the curves γ_{2j}.

Now along any curve γ, $|f(\zeta)| = M$, while the argument of $f(\zeta)$ varies monotonically and tends to $\pm\infty$ at the ends of γ. Hence, by (99) and (100), γ lies outside all the sectors $\Sigma_j(\epsilon)$ for $|\zeta| > \zeta_0(\epsilon)$. Hence each end of γ is asymptotic to one of the rays separating the sectors Σ_j. Thus each Δ_{2j} is "roughly" one of the sectors Σ_{2j} in which f has the asymptotic value ∞. Precisely, if the Δ_{2j} are indexed properly, the ends of γ_{2j} are asymptotic to the two rays bounding Σ_{2j} and

(107a) $$\Sigma_{2j}(\epsilon) \cap \{ |\zeta| > A \} \subset \Delta_{2j}, \qquad A > A(\epsilon),\ \epsilon > 0,$$

and

(107b) $$\Delta_{2j} \cap \{ |\zeta| > B \} \subset \Sigma_{2j}(-\epsilon), \qquad B > B(\epsilon),\ \epsilon > 0,$$

where $\Sigma(-\epsilon)$ is defined by (98).

In a similar fashion it may be shown that the k domains of the ζ-plane corresponding to small neighborhoods of the k finitely-affixed logarithmic branch points of $\mathcal{G}$ are asymptotic to the sectors Σ_{2j+1}.

29. We now consider the structure of $\mathcal{G} \in W^k$ and the corresponding asymptotic behavior of both $f(z)$ and $f'(z)$, where $f(z)$ is given by (91) through (96).

As in §19 we may choose n so that all the logarithmic extensions of $\mathcal{G}$ are "hung," directly or indirectly, onto the first $n+1$ sheets, $S_1, S_2, \cdots, S_{n+1}$, of the basic surface $\mathcal{H} \in W^o$, (see §§17, 18). Let $\mathcal{G}_n$ denote the corresponding surface of class $W_n^{k,+}$; that is, $\mathcal{G}_n$ possesses the same logarithmic extensions as $\mathcal{G}$, but the sheets S_ν, $\nu > n+1$, of $\mathcal{H}$ have been dropped. In §§19–24 we used a sequence of these surfaces $\mathcal{G}_n$, but *now we shall keep n fixed* and use the properties, derived in §§27–28, of the corresponding functions $f_n(\zeta)$ and $f_n'(\zeta)$. Here $f_n(\zeta)$ is given by (54) through (57) with z replaced by ζ. The symbols Σ_j, Δ_j, Δ^*, γ_j and M will have the same meaning as in §§27–28, but are now associated with $f_n(\zeta)$ rather than $f(\zeta)$.

The interval $(b_{n,n}, +\infty)$ of the real ζ-axis corresponds to the interval $(a_n, (-)^{n-1}\infty)$ of the real axis in the sheet S_{n+1} of $\mathcal{G}_n$. Thus $(b_{n,n}, +\infty)$ has just one intersection with a curve γ_{2j}, and, since $(b_{n,n}, +\infty) \subset \Sigma_0$, this curve must be γ_0. Also, by the choice of M, no curve γ_{2j} intersects the segment $[0, b_{n,n}]$ of the real ζ-axis. Let M be large enough so that

(108) $$|a_{n+1}| < M$$

and let $b_{n,n+1}$ be defined as the point of the real ζ-axis corresponding to $w = a_{n+1} \in S_{n+1}$. Then the segment $[b_{n,n+1}, +\infty)$ intersects γ_0 at $\zeta = \beta$, β real.

The complete surface $\mathcal{G}$ is obtained from $\mathcal{G}_n$ by attaching that part, $\mathcal{H}_n$, of the basic surface $\mathcal{H}$ consisting of the sheets

S_{n+2}, $S_{n+3}, \cdots$ to S_{n+1} along the segment $[a_{n+1}, (-)^n\infty)$ in S_{n+1}. As a result, the ζ-plane cut along the segment $[b_{n,n+1}, +\infty)$ corresponds to a part of the whole surface $\mathcal{G}$. We shall denote this domain of the ζ-plane by D_ζ, and the corresponding part of $\mathcal{G}$ by $\mathcal{G}_n'$; $\mathcal{G}_n'$ is $\mathcal{G}_n$ less a cut. Clearly D_ζ contains all of the domains Σ_j, $j\neq 0$, Δ_{2j}, $j\neq 0$, and the curves γ_{2j}, $j\neq 0$. By virtue of (108), $\Delta_0 \cap D_\zeta$ has two components: Δ_0^+ in $\Im\zeta>0$, bounded by γ_0^+, where γ_0^+ consists of that half of γ_0 lying in $\Im\zeta>0$, and the segment $[\beta, +\infty)$ of the real axis, and Δ_0^- in $\Im\zeta<0$, bounded by γ_0^- and the same segment of the real axis; γ_0^- is the reflection of γ_0^+ in the real axis.

30. As noted, $\mathcal{G}$ consists of the two parts $\mathcal{G}_n'$ and $\mathcal{H}_n$. If $f(z)$ is the entire function, as given by Theorem 3, corresponding to $\mathcal{G}$, then the domain D_z of the z-plane, corresponding to $\mathcal{G}_n'$, is bounded by an analytic curve C which is symmetric about the real axis, intersects the real axis at $z=b_{n+1}$ and has both ends at ∞. Also $0\in D_z$. The other domain, H_z, bounded by C corresponds to $\mathcal{H}_n$. The composite function

$$z = \psi(\zeta) = f^{-1}(f_n(\zeta)), \qquad \psi(0) = 0,\ \psi'(0) = 1, \tag{109}$$

maps D_ζ one-one onto D_z. Let the images in D_z of the domains and curves

$$\Sigma_j,\ \Delta_{2j},\ \gamma_{2j},\ (j\neq 0) \text{ and } \Sigma_0^{\pm},\ \Delta_0^{\pm},\ \gamma_0^{\pm} \text{ be}$$

$$T_j,\ D_{2j},\ \Gamma_{2j},\ (j\neq 0) \text{ and } T_0^{\pm},\ D_0^{\pm},\ \Gamma_0^{\pm},$$

respectively. Here Σ_0^+ and Σ_0^- denote the sectors $0<\arg\zeta<\pi/2k$ and $-\pi/2k<\arg\zeta<0$ respectively.

Corresponding to D_{2j}, $j\neq 0$, $\mathcal{G}$ has $k-1$ logarithmic branch points over ∞. $\mathcal{G}$ also has at least one more singularity over ∞, corresponding to the domains $D_0^{\pm}$. Over finite points $\mathcal{G}$ has, of course, the k logarithmic branch points which correspond to T_j, $j=1, 3, \cdots, 2k-1$, and possibly one more singularity.

The nature of these remaining singularities depends on the structure of $\mathcal{H}_n$, that is, on the behavior of the sequence $\{a_\nu\}$ as $\nu\to\infty$.

The curve Γ_0^+ is part of a level curve Λ_0^+ where $|f(z)|=M$. Similarly, the curve Γ_0^- is contained in Λ_0^-. Let $E_0^{\pm}$ be the corresponding domains in which $|f(z)| > M$. We include in $\Lambda_0^+ (\Lambda_0^-)$ all curves bounding $E_0^+ (E_0^-)$; thus Λ_0^+ may have more than one component. Also of course $D_0^+ \subset E_0^+$ and $D_0^- \subset E_0^-$.

31. According to the structure of $\mathcal{G}$ over ∞ there are two cases to consider.

Case I: $\sup_\nu |a_\nu| < \infty$. The only restrictions so far on M are that it satisfy $|a_\nu| < M$, $1 \leqq \nu \leqq n+1$, and $|\alpha_j| < M$, where the α_j are given by (101). We choose M so that

$$|a_\nu| < M, \qquad 1 \leqq \nu < \infty, \tag{110}$$

is also satisfied. Clearly in this case E_0^+ and E_0^- correspond to two logarithmic branch points of $\mathcal{G}$ over ∞, Λ_0^+ is a single curve in $\mathfrak{I}z > 0$, one half being Γ_0^+ and the other lying in H_z. A reflection of E_0^+ and Λ_0^+ in the real axis yields E_0^- and Λ_0^-. In this case the structure of $\mathcal{G}$ over ∞ is the same as that of the surface of class $W_n^{k+1,-}$ obtained by adding a logarithmic extension to $\mathcal{G}_n'$ in place of $\mathcal{K}_n$. In the domain

$$G_z = H_z - (E_0^+ \cup E_0^-) \tag{111}$$

$|f(z)| < M$. We split case I into two subcases depending on the behavior of $f(z)$ in G_z.

Case Ia:

$$\lim_{\nu \to \infty} a_\nu = a \neq \infty.$$

In this case $\mathcal{G}$ has a single singularity over $w = a$ due to the algebraic branch points of $\mathcal{G}$, for just one of the components of $\mathcal{G}$ over $|w - a| < \epsilon$ is infinite sheeted and contains all but a finite number of the algebraic branch points of $\mathcal{G}$. For ϵ sufficiently small the corresponding domain of the z-plane lies in G_z, and $f(z)$ has the asymptotic value a in G_z. This follows easily from I Theorem 5 and the fact that $f(z) \to a$ as $z \to +\infty$, z real. That $f(z) \to a$ is apparent from the structure of $\mathcal{K}_n$ and $a_\nu \to a$.

Case Ib.

$$-\infty < \liminf a_\nu < \limsup a_\nu < \infty.$$

In this case $\mathcal{G}$ has no singularities other than the $2k+1$ logarithmic branch points already noted. For any other singularity would be over a finite value of w, and this is clearly impossible since a_ν does not tend to a limit. Thus $f(z)$ has no paths of determination in G_z. Alternatively, this last statement is a simple consequence of lim inf $a_\nu <$ lim sup a_ν and I Theorem 5.

Case II. lim sup $|a_\nu| = \infty$. Then, no matter how large M is, there exists N such that $|a_N| > M$, and the complete curve in $\mathcal{G}$ over $|w| = M$ which contains the arc corresponding to Γ_0^+ will have no points in the sheets S_ν, $\nu > N$. In consequence Γ_0^- will lead to the same complete curve in $\mathcal{G}$. That is, $\Lambda_0^+ = \Lambda_0^-$, and $E_0^+ = E_0^-$. Considering the structure of $\mathcal{H}_n$ we see that: $E_0 = E_0^+ = E_0^-$ is bounded by 1), a curve containing Γ_0^+ and Γ_0^-, and 2), an infinite number of disjoint Jordan curves in H_z. Hence there are no new unbounded domains in which $f(z)$ is bounded and $\mathcal{G}$ has no finitely affixed singularities other than the k logarithmic branch points already discussed. There is just one more singularity over $w = \infty$ which corresponds to the telescoping family of domains E_0 as $M \to \infty$. This singularity is not a logarithmic branch point but is, so to speak, "a logarithmic branch point complicated with a condensation of algebraic branch points"; in the terminology of Iversen [9] this singularity is directly critical of the second kind.

32. *The sectors T_j corresponding to the sectors Σ_j of D_ζ, $j \neq 0$, and T_0^+, T_0^- are associated with the asymptotic values of both $f(z)$ and $f'(z)$ in the sense of* I §10. *In case I, the domains E_0^+ and E_0^- are associated with the asymptotic values ∞ for both $f(z)$ and $f'(z)$. In case II both f and f' have the asymptotic value ∞ in the subdomains D_0^+ and D_0^- of E_0.*

33. We shall prove these facts and then consider the location of the sectors T_j in the plane.

That $f(z)$ has the asymptotic values α_j in the sectors $T_1, T_3, \cdots, T_{2k-1}$, and the asymptotic value ∞ in the sectors $T_0^+, T_0^-, T_2, T_4, \cdots, T_{2k-2}$, is immediate from (101) and (104), (these equations for f_n, not f), for $\zeta = \psi^{-1}(z)$ maps all the sectors T simultaneously onto linear sectors. To show that $f'(z)$ has the asymptotic value 0 or ∞, according as j is odd or even, in the sector T_j, $1 \leqq j \leqq 2k-1$, and the asymptotic value ∞ in both T_0^+ and T_0^- we use I Lemma 5. Since $\epsilon < \arg \zeta < 2\pi - \epsilon$ is contained in $\epsilon < \arg (\zeta - b_{n,n+1}) < 2\pi - \epsilon$ we have

$$A_1 r^{-2} \sin\frac{\theta}{2} < \left|\frac{dz}{d\zeta}\right| < A_2 \csc^3\frac{\theta}{2}, \qquad r > r_0,\ 0 < \theta < 2\pi,$$

where $\zeta = re^{i\theta}$ and A_1 and A_2 are positive constants. Hence

$$\frac{1}{A_2}\sin^3\frac{\theta}{2}|f_n'(\zeta)| < |f'(z)| < \frac{1}{A_1} r^2 \csc\frac{\theta}{2}|f_n'(\zeta)|,$$

for $r > r_0$. From the left hand inequality it is clear by (100) that $f'(z)$ has the asymptotic value ∞ in T_0^+, T_0^-, T_2, T_4, $\cdots$, T_{2k-2}. Similarly, the right hand inequality together with (99) shows that $f'(z)$ has the asymptotic value 0 in T_1, T_3, $\cdots$, T_{2k-1}. Here of course (99) and (100) apply to $f_n(\zeta)$ rather than $f(\zeta)$.

As for D_0^+, the boundaries of T_0^+ and D_0^+ agree along an infinite arc of C and, since $|f(z)| = M$, but $f(z)$ does not tend to any limit as $z \to \infty$ on Γ_0^+, it is clear that Γ_0^+ runs close enough to the curve separating T_0^+ and T_1 so that we may apply I Lemma 2, and hence both $f(z)$ and $f'(z)$ have the asymptotic value ∞ in D_0^+. Similarly, f and f' have the asymptotic value ∞ in D_0^-.

34. To prove that, in case I, f and f' have the asymptotic value ∞ in E_0^+ and E_0^- requires a different approach, for the mapping function ψ applies only to the subsectors $D_0^\pm$ of $E_0^\pm$. Consider E_0^+; the same procedure will apply to E_0^-. Let $z = \Phi(W)$ map $\Re W > 0$ onto E_0^+ with the points at ∞ corresponding. Since $w = f(z)$ maps E_0^+ onto that part of the Riemann surface of log w lying over $|w| > M$, it follows that

$$f(z) = Me^{i\kappa}e^{W}, \qquad \kappa \text{ a real constant.}$$

Then $f(z) \to \infty$ uniformly as $W \to \infty$ in $|\arg W| \leq \pi/2 - \epsilon$, and $f(z)$ has the asymptotic value ∞ in E_0^+. Also, by I Lemma 5,

$$|f'(z)| = M|e^W| \Big/ \left|\frac{dz}{dW}\right| > \frac{A}{R}\cos^3\Theta e^{R\cos\Theta},$$

for $R > R_0$, where $A > 0$ is a constant and $W = Re^{i\Theta}$. Hence $f'(z)$ has the asymptotic value ∞ in E_0^+, and we have completed the proof of the statements of §32.

35. In case II, E_0 is never simply connected and I §10 does not apply, although, as we have seen, it applies to the subdomains $D_0^\pm$. At any rate, we know from §31 that the sectors $T_1, T_2, \cdots, T_{2k-1}$, together with E_0^+, E_0^-, and possibly G_z in case I, or together with E_0 in case II, represent all the asymptotic tracts of $f(z)$: that is, there is a one-one correspondence between $T_1, \cdots, T_{2k-1}$, etc., and the singularities of $\mathcal{G}$. Although $f'(z)$ has asymptotic values in the sectors $T_1, \cdots, T_{2k-1}$ and also E_0^+, E_0^- (D_0^+, D_0^-) in case I (case II), we have not shown that these represent all the asymptotic tracts of $f'(z)$.

By using (92) we shall show that these sectors do represent all the asymptotic tracts of $f'(z)$ with one possible exception: in some cases $f'(z)$ will have a finite asymptotic value in a tract whose paths lie eventually in G_z, (see §§37, 43). The following procedure will also yield information on the location and nature of the sectors T.

36. Let

$$P(z) = \prod_{\nu=1}^{\infty} E\left(\frac{z}{b_\nu}, q\right), \tag{112}$$

where E is defined by (78) and $\{b_\nu\}$ and q satisfy (93) and (94). Then $P(z)$ is an entire function of order ρ, and $q \leqq \rho \leqq q+1$. Let, as usual,

$$n(t) = \max_{b_\nu \leqq t} \nu, \qquad 0 \leqq t. \tag{113}$$

Then [17 pp. 206–216] $n(t)$ is of order ρ, and in particular,

$$\lim_{t\to\infty} \frac{n(t)}{t^{q+1}} = 0, \tag{114}$$

$$\int_0^\infty \frac{dn(t)}{t^{q+1}} = (q+1)\int_0^\infty \frac{n(t)}{t^{q+2}}\,dt = \sum_{\nu=1}^{\infty} \frac{1}{b_\nu{}^{q+1}} < \infty, \tag{115}$$

and

$$\int_0^\infty \frac{dn(t)}{t^q} = \infty, \qquad \int_0^\infty \frac{n(t)}{t^{q+1}}\,dt = \infty. \tag{116}$$

Also, if $M(r)$ is the maximum modulus of $P(z)$, then [17, p. 216]

$$\log M(r) < Kr^q\left(\int_0^r \frac{n(t)}{t^{q+1}}\,dt + r\int_r^\infty \frac{n(t)}{t^{q+2}}\,dt\right). \tag{117}$$

where K is a constant depending only on q. It follows from (117) that the order of $P(z)$ is $\leqq$ the order of $n(t)$. The following result is weaker, except when $\rho=q+1$, and is a simple consequence of (114), (115), and (117):

$$\log M(r) = o(r^{q+1}), \qquad r \to \infty. \tag{118}$$

Now $P(z) \neq 0$ in $\arg z \neq 0$ and

$$\log P(z) = \int_0^\infty \left\{ \log\left(1 - \frac{z}{t}\right) + \frac{z}{t} + \cdots + \frac{z^q}{qt^q} \right\} dn(t),$$

where both logarithms are determined in $0 < \arg z < 2\pi$ so as to vanish at $z=0$. Integrating by parts and using (114),

$$\log P(z) = \int_0^\infty n(t) \frac{z^{q+1}}{t^{q+1}(z-t)}\, dt, \qquad \arg z \neq 0. \tag{119}$$

If $z = re^{i\theta}$, then taking the real parts in (119) we find

$$\log |P(re^{i\theta})| = A(r, \theta) \cos q\theta - B(r, \theta) \cos (q+1)\theta, \tag{120}$$

where

$$A(r, \theta) = r^{q+2} \int_0^\infty \frac{n(t)dt}{t^{q+1}(t^2 - 2rt \cos \theta + r^2)}, \tag{121}$$

$$B(r, \theta) = r^{q+1} \int_0^\infty \frac{n(t)dt}{t^q(t^2 - 2rt \cos \theta + r^2)}. \tag{122}$$

We shall be concerned with the growth of $A(r, \theta)$ and $B(r, \theta)$ as $r \to \infty$ with $\theta \neq 0$ fixed. For this purpose we introduce

$$A(r) \equiv r^{q+2} \int_0^\infty \frac{n(t)dt}{t^{q+1}(r^2 + t^2)}, \tag{123}$$

and

$$B(r) \equiv r^{q+1} \int_0^\infty \frac{n(t)dt}{t^q(r^2 + t^2)}. \tag{124}$$

Since

$$\frac{1}{2}(r^2 + t^2)(1 - \cos \theta) \leqq r^2 + t^2 - 2rt \cos \theta \leqq 2(r^2 + t^2),$$

we have

$$(125)\qquad A(r,\theta)=\Theta_1 A(r),\qquad B(r,\theta)=\Theta_2 B(r),$$

where

$$(126)\qquad \frac{1}{2}\leqq \Theta_i \leqq \csc^2\frac{\theta}{2},\qquad 0<\theta<2\pi,\quad i=1,2.$$

Thus the growth of $A(r,\theta)$ and $B(r,\theta)$ is essentially the same along all rays $\theta=\text{const.}$ in $0<\delta\leqq\theta\leqq 2\pi-\delta$. For an estimate from above we have

$$(127)\qquad A(r)=o(r^{q+1}),\qquad B(r)=o(r^{q+1}),\qquad r\to\infty.$$

Namely,

$$A(r)\leqq r^q\int_0^r\frac{n(t)dt}{t^{q+1}}+r^{q+2}\int_r^\infty\frac{n(t)dt}{t^{q+3}}$$

and by (114) and (115),

$$A(r)=r^q\cdot o(r)+r^{q+2}\cdot\frac{1}{r}\cdot o(1)=o(r^{q+1}).$$

The second relation of (127) may be proved similarly. Actually $A(r)$ and $B(r)$ are of the same order, ρ, as $n(r)$, but this fact we shall not need, whereas the *little o* in (127) is important. For an estimate from below we need some degree of uniformity: that $A(r)$ is of order ρ is not good enough. For this purpose we introduce

$$(128)\qquad \mu_1(r)\equiv\int_0^r\frac{n(t)dt}{t^{q+1}},\qquad \mu_2(r)\equiv\int_0^r\frac{n(t)dt}{t^q}.$$

It is easily verified, using (114), (115), and (116) that

$$(129)\qquad \mu_1(r)\uparrow\infty,\qquad \mu_2(r)\uparrow\infty,\qquad \textit{as } r\uparrow\infty,$$

$$(130)\qquad \frac{\mu_1(r)}{r}\to 0,\qquad \int_0^\infty\frac{d\mu_1(t)}{t}=\int_0^\infty\frac{\mu_1(t)}{t^2}\,dt<\infty,$$

and

$$(131)\qquad \frac{\mu_2(r)}{r^2}\to 0,\qquad \int_0^\infty\frac{d\mu_2(t)}{t^2}=2\int_0^\infty\frac{\mu_2(t)}{t^3}\,dt<\infty.$$

It is also easy to prove that μ_1 and μ_2 are of orders $\rho-q$ and $\rho-q+1$ respectively. Using (123), (128), and (130), we find

$$A(r) = r^{q+2}\int_0^\infty \frac{d\mu_1(t)}{r^2+t^2} = r^{q+2}\int_0^\infty \mu_1(t)\,\frac{2t\,dt}{(r^2+t^2)^2}$$

$$\geqq r^{q+2}\mu_1(r)\int_r^\infty \frac{2t\,dt}{(r^2+t^2)^2} = \frac{1}{2}\,\mu_1(r)r^q.$$

Treating $B(r)$ in a similar fashion, we have

$$A(r) \geqq \tfrac{1}{2}\mu_1(r)r^q, \qquad B(r) \geqq \tfrac{1}{2}\mu_2(r)r^{q-1}. \tag{132}$$

Collecting the results of (120), (125), (126), (127), and (132), we have

$$\log\,|\,P(z)\,| = o(r^{q+1}), \quad \textit{uniformly in } 0 < \delta \leqq \theta \leqq 2\pi - \delta, \tag{133}$$

and

$$\begin{aligned} A(r,\theta) &\geqq \tfrac{1}{4}\mu_1(r)r^q,\\ B(r,\theta) &\geqq \tfrac{1}{4}\mu_2(r)r^{q-1}. \end{aligned} \tag{134}$$

37. In considering the asymptotic behavior of $f(z)$ and $f'(z) = e^{\pi(z)}P(z)$, where $f(z)$ satisfies (91) through (96), there are three cases to consider:

α) $q < \sigma, \quad c_\sigma > 0, \quad k = \sigma,$

β) $q < \sigma, \quad c_\sigma < 0, \quad k = \sigma - 1,$

γ) $q \geqq \sigma, \quad k = q.$

Case α) and β) are simpler than γ) and we dispose of them first. If $q<\sigma$ we have by (133),

$$\begin{cases} \log\,|\,f'(re^{i\theta})\,| = c_\sigma r^\sigma \cos\sigma\theta + o(r^\sigma),\\ \textit{uniformly in } \delta \leqq \theta \leqq 2\pi - \delta,\ \delta > 0, \end{cases} \tag{135}$$

and by (118),

$$\log\,|\,f'(re^{i\theta})\,| < c_\sigma r^\sigma \cos\sigma\theta + o(r^\sigma), \qquad \textit{all } \theta. \tag{136}$$

We define the linear sectors U by

$$U_j\colon (2j-1)\frac{\pi}{2\sigma} < \arg z < (2j+1)\frac{\pi}{2\sigma}, \quad 0 \leqq j \leqq 2\sigma - 1, \tag{137}$$

and

$$(138)\qquad U_0^+ : 0 < \arg z < \frac{\pi}{2\sigma}; \qquad U_0^- : -\frac{\pi}{2\sigma} < \arg z < 0.$$

It follows from (135) that:

in case α), $f'(z)$ *has the asymptotic value* 0 *in each of the sectors* $U_1, U_3, \cdots, U_{2\sigma-1}$, *and the asymptotic value* ∞ *in each of the sectors* $U_0^+, U_0^-, U_2, U_4, \cdots, U_{2\sigma-2}$, *while*

in case β), $f'(z)$ *has the asymptotic value* 0 *in each of the sectors* $U_0, U_2, \cdots, U_{2\sigma-2}$, *and the asymptotic value* ∞ *in each of the sectors* $U_1, U_3, \cdots, U_{2\sigma-1}$.

In both cases the specified asymptotic values are representative of all the asymptotic tracts of $f'(z)$. *In case* α) U_0^+ *and* U_0^- *arise from the same asymptotic tract of* $f'(z)$. Namely, if there were any more asymptotic tracts of $f'(z)$, then there would exist a path Γ along which $f'(z) \to a$ as $z \to \infty$, and Γ would be asymptotic to one of the rays separating two of the sectors U, that is, ($\arg z$ on Γ) $\to$($\arg z$ on ray) as $z \to \infty$. We shall suppose for the moment that, for case α), this ray is not the positive real axis. If $a \neq \infty$, then an obvious application of I Theorem 7 followed by I Theorem 6 proves that $a=0$ and Γ belongs to the tract represented by the adjacent sector U in which $f'(z) \to 0$. If $a=\infty$, we apply this same argument to $1/f'(z)$, which is holomorphic in a sector about the ray in question since the only zeros of $f'(z)$ occur on the positive real axis. In case α), if Γ is asymptotic to the positive real axis and $|f'(z)| < A$ on Γ, then applying I Theorem 7 to Γ and

$$\arg z = \pi/2\sigma + \epsilon \quad \text{or} \quad \arg z = -\pi/2\sigma - \epsilon$$

we would have $f'(z)$ bounded in U_0, which is impossible. It follows that U_0^+ and U_0^- are connected with a single asymptotic tract of $f'(z)$.

38. In cases α) and β) the sectors T of §32 are approximately the sectors U. Let L_j be the curve separating T_{j-1} from T_j, for $j=2, 3, \cdots, 2k-1$, and let L_1 separate T_0^+ from T_1 and L_{2k} separate T_{2k-1} from T_0^-. That is, by §30,

$$(139)\qquad L_j = \text{image of } \left\{\arg \zeta = (2j-1)\frac{\pi}{2k}\right\} \quad \text{by } z = \psi(\zeta).$$

Then:

in case α), $T_j \sim U_j$, $j=1, 2, \cdots, 2k-1$, *by which is meant:*

$$\arg z \to (2j-1)\pi/2k \qquad \text{as } z \to \infty \text{ on } L_j,\ 1 \leqq j \leqq 2k.$$

Also $f(z)$ *has the same asymptotic value in* U_j *that it has in* T_j: $\alpha_j \neq \infty$ *for* $j=1, 3, \cdots, 2k-1$, *and* ∞ *for* $j=2, 4, \cdots, 2k-2$. *Finally, the corresponding surface* $\mathcal{G}$ *falls into case* II *of* §31, *that is,* $\limsup |a_\nu| = \infty$, *and the sectors* $T_0^{\pm}$ *(or* $D_0^{\pm}$*), in which* $f(z)$ *has the asymptotic value* ∞, *arise from the same asymptotic tract of* $f(z)$.

In case β), $T_j \sim U_{j+1}$, $1 \leqq j \leqq 2k-1$; *that is:* $\arg z \to (2j+1)\pi/2\sigma$ *as* $z \to \infty$ *on* L_j, $1 \leqq j \leqq 2k$. *Also* $f(z)$ *has the same asymptotic value in* U_{j+1} *that it has in* T_j: $\alpha_j \neq \infty$ *for* $j=1, 3, \cdots, 2k-1$, *and* ∞ *for* $j=2, 4, \cdots, 2k-2$. *Finally, the corresponding surface* $\mathcal{G}$ *falls into case* Ia *of* §31, *that is,* $\lim a_\nu = a \neq \infty$, *and* $E_0^+ \sim U_1$, $E_0^- \sim U_{2k+1} = U_{2\sigma-1}$, *and* $G_z \sim U_0$. *Both* $f(z)$ *and* $f'(z)$ *have the asymptotic value* ∞ *in* U_1 *and* $U_{2\sigma-1}$, *while* $f(f')$ *has the asymptotic value* a (0) *in* U_0.

39. *In what follows we shall say that the curve* L *is asymptotic to the ray* $\arg z = \theta_0$ *if and only if* $\arg z \to \theta_0$ *as* $z \to \infty$ *on* L. We shall now prove the results stated in §38. We start by considering case α). Introducing again the ζ-plane by means of (109), it is clear from the asymptotic behavior of $f_n'(\zeta)$ in the sector Σ_j, as given by (99) and (100), that there exist level curves l_j' on which $|f_n'(\zeta)| = 1$ and such that l_j' is asymptotic to the ray, $\arg \zeta = (2j-1)\pi/2k$, which separates Σ_{j-1} and Σ_j, $1 \leqq j \leqq 2k$. It follows from I Lemma 5 that

$$0 < A_1 < |f'(z)| < A_2 |\zeta|^2, \qquad \zeta \in l_j',\ 1 \leqq j \leqq 2k, \tag{140}$$

where A_1 and A_2 are constants. Now from (135), (55), and (57), together with (102) and (103) (which apply to $f_n(\zeta)$), we see by considering the growth of $f(z)$ and $f_n(\zeta)$ along the negative real axes that

$$\lim_{\zeta \to \infty} \frac{|z|^{1+\epsilon}}{|\zeta|} = \infty, \qquad \epsilon > 0, \tag{141}$$

as $\zeta \to \infty$ on the negative real axis. For instance, if $\delta > 0$ and $k = \sigma$ is odd, then for $|\zeta| > r_0$ and $\zeta < 0$,

$$\exp(-|z|^{k+\delta}) < |f(z) - \alpha_k| = |f_n(\zeta) - \alpha_k| < \exp(-|\zeta|^{k-\delta}),$$

and (141) follows. A similar procedure applies when k is even. Now $z=\psi(\zeta)$ is schlicht and $\psi(0)=0$; hence $\psi(\zeta)$ is bounded away from 0 in $0<\theta_0 \leqq \arg(\zeta+1) \leqq 2\pi-\theta_0$. By a simple application of I Theorem 5 to $\zeta/[\psi(\zeta)]^{1+\epsilon}$ we see that (141) *holds uniformly in any sector* $0<\theta_0\leqq \arg\zeta \leqq 2\pi-\theta_0$. Thus by (140)

$$0 < A_1 < |f'(z)| < A_2|z|^3, \qquad z \in L_j'',$$

where L_j' is the image of l_j' by $z=\psi(\zeta)$. It follows from (135) then that L_j' has the ray $\arg z=(2j-1)\pi/2k$ as an asymptote. But l_j'' and $\arg\zeta=(2j-1)\pi/2k$ are asymptotic and it follows from I Lemma 6 that L_j'' and L_j are asymptotic, and hence that L_j is asymptotic to $\arg z=(2j-1)\pi/2k$. It should be pointed out that this proof is not extraneous. Namely, we know to start with that $f'(z)$ has asymptotic values in the sectors T_j and U_j, but this along would not be incompatible with having $\arg z$ oscillate rapidly between $(2j-1)\pi/2k-\epsilon$ and $(2j-1)\pi/2k+\epsilon$ on L_j. This is clear by virtue of I §11.

We already know that $f(z)$ has asymptotic values in the sectors T_j. That $f(z)$ has the same asymptotic values, α_j, in U_j is clear from (135) for $j=1, 3, \cdots, 2k-1$. For j even, when the asymptotic value is ∞, it follows from I Lemma 3 that $f(z)$ has the asymptotic value ∞ in any smaller linear sector contained in U_j, and hence has the asymptotic value ∞ in U_j.

40. To prove that $\mathcal{G}$ falls into case II: if not, $f(z)$ would be bounded on the positive real axis, and applying I Theorem 7 to the sector $0<\arg z<\pi/2k+\epsilon$, $f(z)$ would be bounded in U_0, which is impossible, for we already know (§32) that $f(z)$ has the asymptotic value ∞ in both T_0^+ and T_0^-, and since L_1 and L_{2k} are asymptotic to $\arg z=\pi/2k$ and $\arg z=-\pi/2k$ respectively, T_0^+ and T_0^- are approximately inside U_0^+ and U_0^-. This completes the proof of the proposition of §38 for case α).

41. In case β) the proof is essentially the same except for the proof of the statement commencing with "Finally." To prove that $\mathcal{G}$ falls into case Ia we use (136) in U_0. Since $c_\sigma<0$ it follows that $f(z)$ has a finite asymptotic value a in U_0, and in particular, $f(z)\to a$ as $z\to\infty$ on the positive real axis. Hence we have case Ia. Now Γ_0^+ and Λ_0^+, bounding E_0^+ must lie eventually in

$$\frac{\pi}{2\sigma} - \epsilon < \arg z < \frac{3\pi}{2\sigma} + \epsilon, \tag{142}$$

where ϵ is any positive number. For clearly U_1 and E_0^+ are associated with the same asymptotic tract of $f'(z)$. Also $|f(z)| = M$ on $\Gamma_0^+ \cup \Lambda_0^+$ and M was chosen larger than the modulus of any finite singularity of $\mathcal{G}$. Thus if $\Gamma_0^+ \cup \Lambda_0^+$ had points outside of the sector defined by (142) for arbitrarily large $|z|$, we would have a contradiction, for $f(z)$ has finite asymptotic values in U_0 and U_2.

We must also show that Λ_0^+ lies eventually in

$$\frac{\pi}{2\sigma} - \epsilon < \arg z < \frac{\pi}{2\sigma} + \epsilon$$

for any $\epsilon > 0$, and similarly that Γ_0^+ is asymptotic to $\arg z = 3\pi/2\sigma$. The method we use to prove this depends on the following lemma. *If $\mathcal{G}$ falls into case β) and if $f(z_0)$ is any point of $\mathcal{G}$ such that $|f(z_0)| = M$, then there exists a curve χ on $\mathcal{G}$, joining $f(z_0)$ to $f(0)$ such that*

$$\text{length } (\chi) \leqq A < \infty,$$

where the length of χ is measured, in the metric of the w-plane, on $\mathcal{G}$ and A is a constant depending only on M.

Let Δ: $|w - f(z_0)| < R$ be the maximal schlicht disc in $\mathcal{G}$ with center $f(z_0)$. Since all the finitely-affixed branch points (algebraic and logarithmic) of $\mathcal{G}$ lie over a fixed disc $|w| < R_0$, $R \leqq R_0 + M$ and Δ has at least one of these branch points on its circumference. If there is a logarithmic branch point on this circumference, then there is a fixed path in $\mathcal{G}$ joining it to $f(0)$ and of length $< B$. We then use this path and a radius of Δ and join the two by a spiral of sufficiently small diameter about the logarithmic branch point so that the resulting χ is of length $< R + B + 1 \leqq R_0 + M + B + 1$. If there is an algebraic branch point on the circumference of Δ then we use the radius of Δ ending at this branch point a_ν together with the curve in $\mathcal{G}$

corresponding to the segment $0 \leqq z \leqq b$, of the real z-axis. The resulting χ is of length

$$\leqq R + \int_0^\infty |f'(x)|\, dx, \qquad x \text{ real.}$$

This integral is a finite constant by virtue of (136) and $c_\sigma < 0$. Since $R \leqq R_0 + M$ this completes the proof of the lemma.

If now $\{z_n\}$ were a sequence of points on $\Gamma_0^+ \cup \Lambda_0^+$, the boundary of E_0^+, such that

$$\frac{\pi}{2\sigma} + \epsilon \leqq \arg z_n \leqq \frac{3\pi}{2\sigma} - \epsilon, \qquad \epsilon > 0,$$

and $z_n \to \infty$, then by (135)

$$\log |f'(z)| \geqq \frac{1}{2} |c_\sigma| r^\sigma \sin \frac{\sigma\epsilon}{2}$$

in $|z - z_n| < 1$, for $n > n_0$. *Any* curve χ joining $f(0)$ to $f(z_n)$ would then be of length

$$\geqq \exp \left\{ \frac{1}{2} |c_\sigma| (|z_n| - 1)^\sigma \sin \frac{\sigma\epsilon}{2} \right\}.$$

But this expression tends to ∞ as $n \to \infty$, and, since $|f(z_n)| = M$, we would have a contradiction of our lemma. This shows then that $f(z)$ has the asymptotic value ∞ in U_1 and that $E_0^+ \sim U_1$. The same procedure works on $U_{2\sigma-1}$ and E_0^-. This completes the proof of §38.

42. We are left with case γ), $q \geqq \sigma$, $k = q$, to consider. As we shall show, the curves L_j may have definite wobbles in this case and not be asymptotic to any ray. Our starting point here is (120) and (134). We obviously look for the linear sectors of the z-plane in which $\cos q\theta$ and $\cos (q+1)\theta$ have opposite signs.

If $q = 0$ there is just one such sector:

$$(143) \qquad V_0\colon \ \cos 0 = 1 > 0, \quad \cos \theta < 0, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

If $q>0$ then the sectors in which $\cos q\theta \cos (q+1)\theta<0$ are V_j, $0\leqq j\leqq 2q$, where

$$(144)\qquad \begin{cases} V_j\colon \dfrac{(2j+1)\pi}{2(q+1)}<\theta<\dfrac{(2j+1)\pi}{2q}, & 0\leqq j\leqq q-1, \\ V_q\colon \dfrac{(2q+1)\pi}{2(q+1)}<\theta<\dfrac{(2q+3)\pi}{2(q+1)}, & \\ V_j\colon \dfrac{(2j-1)\pi}{2q}<\theta<\dfrac{(2j+3)\pi}{2(q+1)}, & q+1\leqq j\leqq 2q. \end{cases}$$

Also

$$(145)\qquad (-)^j \cos q\theta>0, \quad (-)^{j+1}\cos (q+1)\,\theta>0, \qquad \text{in } V_j,\ 0\leqq j\leqq 2q.$$

The sectors V_j and V_{2q-j} are switched by a reflection in the real axis. As for the sectors between the V's, let V_j^* be the sector between V_{j-1} and V_j, $1\leqq j\leqq 2q$, and let V_0^* be the sector between V_{2q} and V_0. Then

$$(146)\qquad (-)^j \cos q\theta>0, \quad (-)^j\cos (q+1)\theta>0, \qquad \text{in } V_j^*,\ 0\leqq j\leqq 2q.$$

If ω_j is the angular opening of V_j and ω_j^* that of V_j^*, then

$$(147)\qquad \omega_j<\frac{\pi}{q+1},\ j\neq q, \quad \text{and} \quad \omega_q=\frac{\pi}{q+1},$$

and

$$(148)\qquad \omega_j^*<\frac{\pi}{q+1},\ j\neq 0, \quad \text{and} \quad \omega_0^*=\frac{\pi}{q+1}.$$

Finally, if V is the sector $\alpha<\theta<\beta$, we denote the sector $\alpha+\epsilon<\theta<\beta-\epsilon$ by $V(\epsilon)$.

43. *For the sake of simplicity we shall assume that* $q>0$. Since $q\geqq\sigma$,

$$\log |e^{\pi(z)}| = O(|z|^{\sigma}) = O(|z|^{q}),$$

and we have, by (120), (129), and (134),

$$(149)\qquad \begin{cases} \log |f'(re^{i\theta})| \geqq \dfrac{q\epsilon}{4\pi}\mu_1(r)r^q, & \\ re^{i\theta} \in V_j(\epsilon), \quad r \geqq r_0, & j = 0, 2, 4, \cdots, 2q, \end{cases}$$

and

$$(150)\qquad \begin{cases} \log |f'(re^{i\theta})| \leqq -\dfrac{q\epsilon}{4\pi}\mu_1(r)r^q, & \\ re^{i\theta} \in V_j(\epsilon), \quad r \geqq r_0, & j = 1, 3, 5, \cdots, 2q-1. \end{cases}$$

Hence $f'(z)$ has the asymptotic value ∞ (0) in V_j for j even (odd). Using I Theorems 7 and 6, (147), and (148), it may be readily proved, in a fashion similar to §37, that $f'(z)$ has no other asymptotic values with the possible exception of a finite asymptotic value occurring in part of V_0^*. Comparing this result with §32 we see that: V_j and T_j are associated with the same asymptotic tracts of $f'(z)$, $1 \leqq j \leqq 2q-1$, and V_0 and V_{2q} are related to D_0^+ and D_0^-, respectively, in the same fashion.

44. Let the curves L_j separating the T_j be given by (139). We show next that L_j eventually stays inside $V_j^*(-\epsilon)$, where ϵ is any positive constant. The method is that of §39, using the same l_j' and L_j'. Equation (140) is still valid. Using (120), (130), (134), (55), (57), and the fact that $f'(z)$ is real on the negative real axis, we have, in place of (141),

$$(151)\qquad \lim_{\zeta\to\infty} \frac{|z|^{q+1+\epsilon}}{|\zeta|^q} = \infty, \qquad \epsilon > 0,$$

uniformly in $0 < \delta \leqq \arg \zeta \leqq 2\pi - \delta$. Thus we have from (140)

$$0 < A_1 < |f'(z)| < A_2|z|^4, \qquad z \in L_j', \ 1 \leqq j \leqq 2q.$$

By comparing this with (149) and (150) we see that L_j' runs inside $V_j^*(-\epsilon)$ for $|z|$ sufficiently large. Now l_j' is asymptotic to $\arg \zeta = (2j-1)\pi/2k$. It follows from I Lemma 6 that for $|z| > r_0$, $z \in L_j$, there exists $z' \in L_j'$ such that $|z'| > r_0/2$ and $|\arg z - \arg z'| < \epsilon$. Thus L_j runs inside $V_j^*(-2\epsilon)$ for $|z|$ sufficiently large.

45. To summarize, *in case* γ), *if* $q>0$, *then there exists* $r(\epsilon)$ *for* $\epsilon>0$ *such that*

$$V_j(\epsilon) \cap \{|z| > r(\epsilon)\} \subset T_j, \qquad 1 \leqq j \leqq 2q-1.$$

Also, $f(z)$ *and* $f'(z)$ *have the same asymptotic values in* V_j *that they have in* T_j, $1 \leqq j \leqq 2q-1$. This last property of $f(z)$ follows easily from I Lemma 3.

If $q=0$, $V_0(\epsilon)$ *lies eventually inside* T_0 *and* f *and* f' *both have the asymptotic value* ∞ *in* V_0 *and* T_0.

As to the oscillation of arg z on L_j and the question of how V_j^* is divided between T_{j-1} and T_j, all extremes may arise. Namely, *for any given* $q>0$ *there exist three functions* $f_1(z)$, $f_2(z)$, *and* $f_3(z)$ *of the form* (91) *through* (95), *with* $\sigma \leqq q$, *such that:*

1° *For* $f_1(z)$ *the sectors* T_j, $1 \leqq j \leqq 2k-1$, *are asymptotic to the sectors in which* $\cos q\theta$ *has a single sign, that is,* L_j *is asymptotic to* $\arg z = (2j-1)\pi/2q$, $1 \leqq j \leqq 2q$, *and both* $f_1(z)$ *and* $f_1'(z)$ *have the same asymptotic values in*

$$(2j-1)\frac{\pi}{2q} < \arg z < (2j+1)\frac{\pi}{2q}$$

that they have in T_j, $1 \leqq j \leqq 2q-1$.

2°. *For* $f_2(z)$ *the sectors* T_j, $1 \leqq j \leqq 2k-1$, *are asymptotic to the sectors in which* $\cos(q+1)\theta$ *has a single sign, that is,* L_j *is asymptotic to* $\arg z = (2j+1)\pi/2(q+1)$, $1 \leqq j \leqq 2q$, *and both* $f_2(z)$ *and* $f_2'(z)$ *have the same asymptotic values in*

$$(2j+1)\frac{\pi}{2(q+1)} < \arg z < (2j+3)\frac{\pi}{2(q+1)}$$

that they have in T_j, $1 \leqq j \leqq 2q-1$.

3°. *For* $f_3(z)$ *the sector* T_j *contains no linear sector larger than* V_j. *That is,*

$$\limsup_{z\to\infty,\, z\in L_j} \arg z = \frac{(2j+1)\pi}{2(q+1)}, \qquad 1 \leqq j \leqq q, \tag{152a}$$

and

$$\liminf_{z\to\infty, z\in L_j} \arg z = \frac{(2j-1)\pi}{2q}, \qquad 1 \leqq j \leqq q, \tag{152b}$$

while similar relations hold for $q+1 \leqq j \leqq 2q$. *The property expressed by* (152) *may also be stated as follows: for any* $\epsilon > 0$, L_j *contains infinitely many arcs which join the two rays bounding* $V_j^*(\epsilon)$ *and all but a finite number of these arcs are contained in* $|z| > r_0$, *for any* $r_0 > 0$.

46. In constructing these three functions we will take $\sigma = 0$ although this is not essential.

For $f_1(z)$ we may use any function given by $f_1'(z) = P(z)$, $P(z)$ being given by (112), such that

$$\lim_{r\to\infty} \frac{A(r)}{B(r)} = \infty, \tag{153}$$

where $A(r)$ and $B(r)$ are defined by (123) and (124). For by (120), (125), (126), (134), and (153) we have

$$\log |P(re^{i\theta})| \geqq \tfrac{1}{2}A(r) \cos q\theta - \eta_1, \qquad \cos q\theta > 0,$$

and

$$\log |P(re^{i\theta})| \leqq \tfrac{1}{2}A(r) \cos q\theta + \eta_2, \qquad \cos q\theta < 0,$$

where η_1 and η_2 are uniformly $o(A(r))$ in $0 < \epsilon \leqq \theta \leqq 2\pi - \epsilon$. Replacing (149) and (150) by these two inequalities and then proceeding as before, with the method of §44, it is readily proved that $f_1(z)$ has the property 1°.

In a similar fashion, any $f_2(z)$ given by $f_2'(z) = P(z)$, which is such that

$$\lim_{r\to\infty} \frac{A(r)}{B(r)} = 0, \tag{154}$$

will have the property 2°.

Finally, any $f_3(z)$ given by $f_3'(z) = P(z)$ which is such that

$$\liminf_{r\to\infty} \frac{A(r)}{B(r)} = 0, \qquad \limsup_{r\to\infty} \frac{A(r)}{B(r)} = \infty, \tag{155}$$

will have the property 3°. For, taking account of §45, it will be sufficient to show that

$$\limsup_{z\to\infty, z\in L_j} \arg z \geqq \frac{(2j+1)\pi}{2(q+1)}, \qquad 1 \leqq j \leqq q, \tag{156}$$

in order to prove (152a). Now, if (156) were false for a given j, then there would exist a linear sector $V' \subset V_j^*$ such that

$$V' \cap \{|z| > r_0\} \subset T_j.$$

By I Lemma 3, $f_3'(z)$ would possess the same asymptotic value in V' that it has in T_j. But $\cos q\theta$ and $\cos (q+1)\theta$ have the same sign in V_j^*, and by (155), (120), (125), and (126) there exist two sequences of points, $\{z_n\}$ and $\{z_n'\}$, on the bisector of V' such that as $n\to\infty$, $z_n\to\infty$, $z_n'\to\infty$, $f_3'(z_n)\to 0$, and $f_3'(z_n')\to\infty$. Thus $f_3'(z)$ can have no asymptotic value in V'. Therefore the assumption that (156) is false leads to a contradiction. The proof of (152b) is similar.

47. We are left then with the problem of constructing three examples for each $q>0$, one satisfying (153), one (154), and one (155).

We note first that if $m(t)\geqq 0$ is any increasing function, defined for $t\geqq 0$, such that

$$\int_0^\infty \frac{m(t)dt}{t^{q+2}} < \infty, \qquad \int_0^\infty \frac{m(t)dt}{t^{q+1}} = \infty, \tag{157}$$

then $n(t)$, defined by $n(t) = [m(t)]$, satisfies (115) and (116). Also it is easily verified that

$$\int_0^\infty \frac{n(t)dt}{t^{q+\eta}(r^2+t^2)} \sim \int_0^\infty \frac{m(t)dt}{t^{q+\eta}(r^2+t^2)}, \qquad r\to\infty, \tag{158}$$

for $\eta = 0$ and $\eta = 1$.

48. Example of 1°. Set

$$m(t) = \begin{cases} 0, & 0 \leqq t < 1 \\ t^q, & 1 \leqq t. \end{cases}$$

Then $m(t)$ satisfies (157), and by (158)

$$A(r) \sim r^{q+2} \int_1^\infty \frac{dt}{t(r^2+t^2)} = \frac{1}{2} r^q \log(1+r^2),$$

$$B(r) \sim r^{q+1} \int_1^\infty \frac{dt}{r^2+t^2} = r^q\left(\frac{\pi}{2} - \operatorname{Tan}^{-1}\frac{1}{r}\right) \sim \frac{1}{2}\pi r^q,$$

and (153) obviously follows.

49. Example of 2°. Set

$$m(t) = \begin{cases} 0, & 0 \leqq t < e \\ t^{q+1}(\log t)^{-2}, & e \leqq t. \end{cases}$$

Then $m(t)$ is increasing and satisfies (157). Also, by (158),

$$A(r) \sim r^{q+2} \int_e^\infty \frac{dt}{(r^2+t^2)(\log t)^2}$$
$$< r^q \int_e^r \frac{dt}{(\log t)^2} + r^{q+2} \int_r^\infty \frac{dt}{t^2(\log t)^2}.$$

Now

$$\int_e^r \frac{dt}{(\log t)^2} < \int_e^{\sqrt{r}} dt + \int_{\sqrt{r}}^r \frac{dt}{(\log \sqrt{r})^2} = O(r(\log r)^{-2}),$$

and

$$\int_r^\infty \frac{dt}{t^2(\log t)^2} < \int_r^\infty \frac{dt}{t^2(\log r)^2} = O(r^{-1}(\log r)^{-2}).$$

Therefore

$$A(r) = O(r^{q+1}(\log r)^{-2}), \qquad r \to \infty.$$

On the other hand,

$$B(r) \sim r^{q+1} \int_e^\infty \frac{tdt}{(r^2+t^2)(\log t)^2} > \frac{1}{2} r^{q+1} \int_r^\infty \frac{dt}{t(\log t)^2}$$
$$= \frac{r^{q+1}}{2\log r}.$$

Consequently

$$\frac{A(r)}{B(r)} = O\left(\frac{1}{\log r}\right), \qquad r \to \infty,$$

and (154) holds.

50. Example of 3°. Let $\{\lambda_n\}$ be a sequence of positive numbers such that

$$1 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \tag{159}$$

and define μ_n and a_n by

$$\mu_n = 2^n \lambda_n, \qquad n \geqq 1, \tag{160}$$

and

$$a_n = \mu_n^{(n-1)/n} = 2^{n-1} \lambda_n^{(n-1)/n}, \qquad n \geqq 1. \tag{161}$$

We choose $\{\lambda_n\}$ so that the following four conditions are satisfied:

$$\lambda_n < \mu_n < \mu_n^2 < \lambda_{n+1}, \qquad n \geqq 1, \tag{162}$$

$$a_{n-1} < \lambda_n^{(n-1)/n}, \qquad n > 1, \tag{163}$$

$$\sum_{\nu=n+1}^{\infty} \nu \lambda_\nu^{-1/\nu} < \frac{a_n}{\mu_n^2}, \qquad n \geqq 1, \tag{164}$$

$$\sum_{\nu=n+1}^{\infty} \frac{1}{\lambda_\nu} < \frac{1}{\lambda_n^2}, \qquad n \geqq 1. \tag{165}$$

That such a choice of $\{\lambda_n\}$ is possible is not difficult to prove; for example, it is easy to show that all four of these properties follow from the single condition.

$$\lambda_{n+1}^{1/(n+1)} > (n+1) 2^{n+1} \lambda_n^{(n+1)/n}, \qquad n \geqq 1.$$

Let $m(t)$ be defined by: $m(t) = 0$, for $0 \leqq t < \lambda_1 = 1$, and

$$m(t) = \begin{cases} t^{q+1-1/n}, & \lambda_n \leqq t \leqq \mu_n,\ n \geqq 1, \\ a_n t^q, & \mu_n < t < \lambda_{n+1},\ n \geqq 1. \end{cases} \tag{166}$$

By (162) this definition makes sense, and by (161) $m(t)$ is continuous at the points $t = \mu_n$, $n \geqq 1$. At the discontinuities of $m(t)$ we have by (163)

$$m(\lambda_n - 0) = a_{n-1}\lambda_n^{q} < \lambda_n^{q+1-1/n} = m(\lambda_n + 0),$$

for $n > 1$. Thus $m(t)$ is increasing. Since $m(t) \leqq M(t)$ where $M(t) = 0$ for $0 \leqq t < \lambda_1$ and

$$M(t) = t^{q+1-1/n}, \quad \lambda_n \leqq t < \lambda_{n+1},\ n \geqq 1,$$

we have

$$\int_0^\infty \frac{m(t)dt}{t^{q+2}} \leqq \int_0^\infty \frac{M(t)dt}{t^{q+2}} = \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^{-1-1/n}dt$$

$$< \sum_{n=1}^{\infty} n\lambda_n^{-1/n} < \infty,$$

by (164). Also, for $t \geqq 1$, $m(t) \geqq t^q$, and therefore the conditions (157) are satisfied. Now, by (158), as $n \to \infty$,

$$A(\lambda_n) \sim \lambda_n^{q+2} \int_0^\infty \frac{m(t)dt}{t^{q+1}(\lambda_n^2 + t^2)} \leqq \lambda_n^{q+2} \int_0^\infty \frac{M(t)dt}{t^{q+1}(\lambda_n^2 + t^2)}$$

$$< \lambda_n^{q} \int_1^{\lambda_n} t^{-1/(n-1)} dt + \lambda_n^{q+2} \sum_{\nu=n}^{\infty} \int_{\lambda_\nu}^{\infty} t^{-2-1/\nu} dt$$

$$= O(\lambda_n^{q+1-1/(n-1)}) + O(\lambda_n^{q+1-1/n}) + O\left(\lambda_n^{q+2} \sum_{\nu=n+1}^{\infty} \lambda_\nu^{-1}\right).$$

Using (165) we have

$$(167) \qquad A(\lambda_n) = O(\lambda_n^{q+1-1/n}), \qquad n \to \infty.$$

On the other hand,

$$B(\lambda_n) \sim \lambda_n^{q+1} \int_0^\infty \frac{m(t)dt}{t^{q}(\lambda_n^2 + t^2)} > \frac{1}{2} \lambda_n^{q+1} \int_{\lambda_n}^{\mu_n} \frac{m(t)dt}{t^{q+2}}$$

$$= \frac{n}{2} \lambda_n^{q+1} (\lambda_n^{-1/n} - \mu_n^{-1/n}) = \frac{n}{4} \lambda_n^{q+1-1/n}.$$

Comparing this result with (167), we have

$$\liminf_{r\to\infty} \frac{A(r)}{B(r)} = 0. \tag{168}$$

In the opposite direction, we consider $r=\mu_n^2$. Then

$$A(\mu_n^2) \sim \mu_n^{2q+4}\int_0^\infty \frac{m(t)dt}{t^{q+1}(\mu_n^4+t^2)} > \frac{1}{2}\mu_n^{2q}\int_{\mu_n}^{\mu_n^2}\frac{m(t)dt}{t^{q+1}},$$

and by (166) and (162) we have

$$A(\mu_n^2) > \tfrac{1}{3}a_n\mu_n^{2q}\log\mu_n. \tag{169}$$

Also, $m(t)\leqq a_n t^q$ for $t<\lambda_{n+1}$, and

$$\begin{aligned} B(\mu_n^2) &\sim \mu_n^{2q+2}\int_0^\infty \frac{m(t)dt}{t^q(\mu_n^4+t^2)} \\ &< a_n\mu_n^{2q-2}\int_0^{\mu_n^2} dt + a_n\mu_n^{2q+2}\int_{\mu_n^2}^{\lambda_{n+1}}\frac{dt}{t^2} \\ &\quad + \mu_n^{2q+2}\sum_{\nu=n+1}^{\infty}\int_{\lambda_\nu}^{\lambda_{\nu+1}}\frac{M(t)dt}{t^{q+2}} \\ &< a_n\mu_n^{2q} + a_n\mu_n^{2q} + \mu_n^{2q+2}\sum_{\nu=n+1}^{\infty}\nu\lambda_\nu^{-1/\nu}. \end{aligned}$$

Then by (164)

$$B(\mu_n^2) = O(a_n\mu_n^{2q}),$$

and comparing this result with (169),

$$\limsup_{r\to\infty}\frac{A(r)}{B(r)} = \infty. \tag{170}$$

But (168) and (170) constitute (155) and the proof is complete.

REFERENCES

1. L. Bieberbach, "Über einen Satz des Herrn Carathéodory," *Göttinger Nachrichten*, 1913, pp. 552–560.
2. L. Bieberbach, "Lehrbuch der Funktionentheorie," Bd. II, 2$^{\text{end}}$ ed., Leipzig, 1931.
3. C. Carathéodory, "Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten," *Math. Ann.* v. 72 (1912), pp. 107–144.
4. C. Carathéodory, "Conformal Representation," Cambridge, 1941.
5. G. Elfving, "Über eine Klasse Riemannschen Flächen und ihre Uniformisierung," *Acta Soc. Sci. Fenn.*, N.s. A, vol. II, no. 3.
6. G. Elfving, "Über Riemannschen Flächen und Annäherung von meromorphen Funktionen," *8 Congr. Math. scand.* Stockholm, 1934, pp. 96–105.
7. L. R. Ford, "Automorphic Functions," New York, 1929.
8. G. H. Hardy and E. M. Wright, "An introduction to the theory of numbers," Oxford, 1938.
9. F. Iversen, "Recherches sur les fonctions inverses des fonctions méromorphes," thesis, Helsingfors, 1914.
10. E. Lindwart and G. Polya, "Über einen Zusammenhang zwischen der Konvergenz von Polynomfolgen und der Verteilung ihrer Wurzeln," *Rend. Palermo*, v. 37 (1914), pp. 297–304.
11. J. E. Littlewood, "Lectures on the theory of functions," Oxford, 1944.
12. G. R. Mac Lane, "Concerning the uniformization of certain Riemann surfaces allied to the inverse-cosine and inverse-gamma surfaces," *Trans. Am. Math. Soc.*, v. 62 (1947), pp. 99–113.
13. G. R. Mac Lane, "Riemann surfaces and asymptotic values associated with certain real entire functions," *Proc. Internat. Congr. Math., Cambridge 1950*, v. 1, p. 396.
14. P. Montel, "Familles normales," Paris, 1927.
15. R. Nevanlinna, "Über die Herstellung transzendenter Funktionen als Grenzwerte rationaler Funktionen," *Acta Math.*, v. 55 (1930), pp. 259–276.
16. R. Nevanlinna, "Über Riemannsche Flächen mit endlich vielen Windungspunkten," *Acta Math.*, vol. 58 (1932), pp. 295–373.
17. R. Nevanlinna, "Eindeutige analytische Funktionen," Berlin, 1936.
18. N. Obrechkoff, "Quelques classes de fonctions entières limites de polynomes et de fonctions méromorphes limites de fractions rationnelles," *Act. Sci. et Ind.*, n. 891, Paris, 1941.
19. W. F. Osgood, "Lehrbuch der Funktionentheorie," v. 1, 5th ed., Leipzig, 1928.
20. G. Polya, "Über Annäherung durch Polynome mit lauter reellen Wurzeln," *Rend. Palermo*, v. 36 (1913), pp. 279–295.

21. G. **Polya** and G. **Szegö**, "Aufgaben und Lehrsätze aus der Analysis," v. 2, Berlin 1925.
22. T. **Rado**, "Sur la représentation conforme de domaines variables," *Acta Szeged*, v. 1 (1922–23), pp. 180–186.
23. T. **Rado**, "Bemerkung zu einem Unitätssatze der konformen Abbildung," *Acta Szeged*, v. 1 (1922–23), pp. 101–103.
24. E. C. **Titchmarsh**, "Theory of functions," 2nd ed., Oxford 1939.
25. G. **Valiron**, "Integral functions," Toulouse, 1923.
26. L. I. **Volkovyskii**, "Convergent sequences of Riemann Surfaces" (in Russian) *Mat Sb.* N.S. 23(65), (1948), pp. 361–382. See *Math. Rev.* v. 10.
27. L. I. **Volkovyskii**, "The determination of the type of certain classes of simply-connected Riemann surfaces," (in Russian) *Mat. Sb.* N.S. 23 (65), (1948), pp. 229–258. See *Math. Rev.* v. 10.
28. J. L. **Walsh**, "Interpolation and approximation by rational functions in the complex domain," New York, 1935.